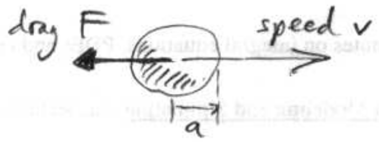
MATH 46 WORKSHEET: Dimensional Analysis. W 3/26/08

Say we suspect that drag force F depends only on a sphere's radius a, its speed v, and the surrounding fluid density ρ.

$$\begin{array}{c} \\ M \\ L \\ T \end{array} \begin{array}{c} a \quad v \quad \rho \quad F \\ \left[\begin{array}{cccc} & & & \\ & & & \\ & & & \end{array} \right] \end{array}$$



a) Fill in the matrix with the dimensions. [If you want to work out dimensions of F, ask your neighbor about Newton's 2nd Law!]

b) Find a dimensionless combination of a, v, ρ, F. Call it $\pi = \ldots$

c) Give the vector of powers, ie $\vec{\alpha} = [\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4]$ such that $\pi = a^{\alpha_1} v^{\alpha_2} \rho^{\alpha_3} F^{\alpha_4}$:

$\vec{\alpha} = [\qquad\qquad\qquad]$

Is this choice unique?
What is the form of all such vectors?
What subspace of the 3×4 matrix do they lie in?

d) Can there be another dimensionless combo. (linearly indep. of $\vec{\alpha}$)? Use linear algebra to prove your claim [Hint: uses matrix rank]

e) So what does Buckingham Pi Theorem tell you?
How must F depend on a, v, ρ? $F = \ldots$

this is ~~Stokes' law~~ sorry, actually inertial drag

f) If F also depended on viscosity η (units $ML^{-1}T^{-1}$) repeat part e). [use the back].

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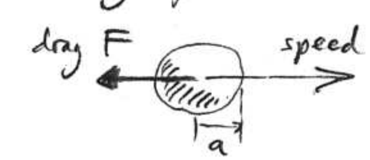
Say we suspect that drag force F depends only on a sphere's radius a, its speed v, and the surrounding fluid density ρ.

$$\begin{array}{c|cccc} & a & v & \rho & F \\ \hline M & & & 1 & 1 \\ L & 1 & 1 & -3 & 1 \\ T & & -1 & & -2 \end{array}$$

"$F = ma$" $= MLT^{-2}$



the key is to recognize you're searching for lin. combo's of columns which gives $\begin{bmatrix}0\\0\\0\end{bmatrix}$, ie $A\vec{\alpha} = \vec{0}$

a) Fill in the matrix with the dimensions. [If you want to work out dimensions of F, ask your neighbor about Newton's 2nd Law!]

b) Find a dimensionless combination of a, v, ρ, F. Call it $\pi = \ldots$

$$\frac{F}{v^2\rho a^2} \quad \text{or} \quad \frac{v^2\rho a^2}{F} \quad \text{or} \quad \left(\frac{v^2\rho a^2}{F}\right)^k \text{ for any } k$$

for this case

c) Give the vector of powers, ie $\vec{\alpha} = [\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4]$ such that $\pi = a^{\alpha_1} v^{\alpha_2} \rho^{\alpha_3} F^{\alpha_4}$:

$$\vec{\alpha} = [-2\ -2\ -1\ \ 1]$$

Is this choice unique? no (see 2 examples in b).

What is the form of all such vectors? $k\vec{\alpha}$ where $k \in \mathbb{R}$ is scalar.

What subspace of the 3x4 matrix do they lie in? Nul A

Note $\vec{\alpha}$ really is a column vector in $\mathbb{R}^4$. Nullspace $A = \{\vec{\alpha} : A\vec{\alpha} = \vec{0}\}$

d) Can there be another dimensionless combo (linearly indep. of $\vec{\alpha}$)? Use linear algebra to prove your claim [Hint: use matrix rank]

rank $A = 3$ since there's 3 pivots when row-reduce. $\dim \text{Nul } A = m - \text{rank } A = 4-3 = 1$

$\Rightarrow$ no more lin. indep. $\vec{\alpha}$'s.

e) So what does Buckingham Pi Theorem tell you? Tells you $\pi_1 = $ const.

How must F depend on a, v, ρ? $F = c\rho a^2 v^2$ ← inertial drag (low viscosity limit)

unknown const.

this is harder, but get $F = c\rho a^2 v^2 g(\pi_2)$ where $\pi_2 = \frac{\eta}{\rho v a} =$ "Reynolds #". ← this is ~~Stokes' law~~

f) If F also depended on viscosity η (units $ML^{-1}T^{-1}$) repeat part e) [use the back]