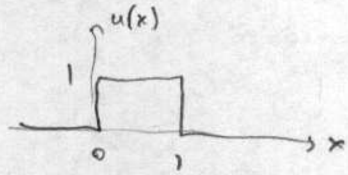
MATH 46 WORKSHEET : Convolution

5/25/07
Barnett

Work out the convolution of $u(x)$ and $v(x)$:

A) $$u(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$



$v(x)$ = same

B) $u(x)$ = anything

$v(x) = \delta(x)$ delta 'function' (distribution)

C) $u(x) = v(x) = e^{-\frac{x^2}{2}}$ Gaussian.

[Hint : complete the square then use Gaussian integral]

How wide is the answer compared to original Gaussian?

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Work out the convolution of $u(x)$ and $v(x)$:

see "The Joy of Convolution" web applet to visualize better.

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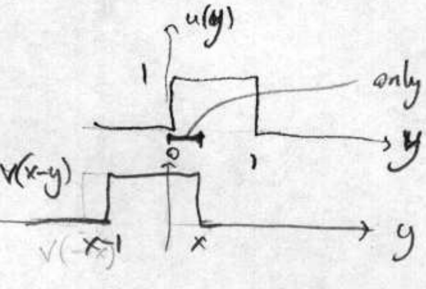
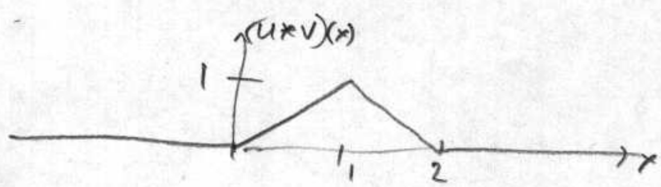
$v(x)$ = same



$$(u * v)(x) = \int_{-\infty}^{\infty} v(x-y)\, u(y)\, dy = \begin{cases} 0 & , \; x < 0 \text{ or } x > 2 \\ x & , \; 0 < x < 1 \\ 2-x & , \; 1 < x < 2 \end{cases}$$



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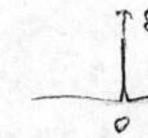
$v(x) = \delta(x)$ delta 'function' (distribution)



$$(u*v)(x) = \int_{-\infty}^{\infty} \delta(x-y)\, u(y)\, dy = u(x)$$ by 'sifting property' of δ.

get the same function back (convolution w/ δ has no effect).

C) $u(x) = v(x) = e^{-\frac{x^2}{2}}$ Gaussian.

[Hint : complete the square then use Gaussian integral]

$$-y^2 + xy - \frac{x^2}{2} = -\left(y - \frac{x}{2}\right)^2 + \frac{x^2}{4} - \frac{x^2}{2}$$

$$(u*v)(x) = \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2}} e^{-y^2/2}\, dy = \int e^{\overbrace{-\frac{x^2}{2} + xy - \frac{y^2}{2} - \frac{y^2}{2}}}\, dy$$

$$= \int e^{-(y-x/2)^2 - x^2/4}\, dy = e^{-x^2/4} \cdot \int e^{-y^2}\, dy = \sqrt{\pi}\, e^{-x^2/4}$$

changing var for y

Gaussian with twice the variance

its width is $\sqrt{2}$ times as big $\leftarrow$ How wide is the answer compared to original Gaussian?