Math 46, Applied Math (Spring 2011): Midterm 1

2 hours, 50 points total, 6 questions worth varying number of points

1. [9 points] A fluid of density ρ rests in a gravitational field strength g (units of acceleration). Surface waves at a frequency f (units of inverse time) may propagate in the fluid, and have wavelength λ. Let us first assume ρ, g, λ and f (and only these variables) are related by a physical law.

   (a) How many (independent) dimensionless quantities are there? Give them. [Hint: a dimensions matrix will help].

   (b) Write the most specific formula you can for how the frequency f must depend on the other three parameters.

   (c) If now a fifth parameter, the surface tension s (units mass per time squared), is also involved in the physical law, use the Buckingham Pi Theorem to deduce the most specific formula for how f must depend on the other four parameters. [Hint: in your answer, f must only appear once.]

2. [11 points] A mass released from rest on an aging spring is described by the model

$$my'' = -ke^{-at}y, \qquad y(0) = L, \qquad y'(0) = 0,$$

where the dynamical variable $y(t)$ is the displacement of the mass vs time.

(a) What are the possible timescales ? [Hint: a dimensions matrix will help]

(b) Choosing an appropriate timescale to give a nonsingular problem in the limit of small aging rate a, and a lengthscale, non-dimensionalize the problem, and give the resulting *small* parameter ε:

[Hint: don't forget the ICs] Answer for $\varepsilon =$

(c) One choice of timescale results in the following non-dimensionalized IVP,

$$y'' = -\frac{1}{\varepsilon^2} e^{-t} y, \qquad\qquad y(0) = 1, \qquad y'(0) = 0.$$

Find the WKB approximation to the solution to this IVP (give your answer in terms of ε and elementary functions only):

[BONUS: until roughly what time t do you expect this to be accurate?]

3. [5 points] Find the leading order perturbation approximation of all roots of $\varepsilon x^4 - x + 1 = 0, \quad \varepsilon \ll 1.$

4. [7 points] Consider the following IVP, where ε is a small parameter,

$$y' = \frac{y}{1+\varepsilon y}, \qquad\qquad y(0) = 1.$$

(a) Use a perturbation expansion to find a 2-term approximation:

(b) Find the residual function of the *unperturbed* solution. Is it uniformly convergent to zero as $\varepsilon \to 0$, on $t \in (0, \infty)$?

5. [9 points] Use singular perturbation methods to find a uniform approximate solution to the boundary-value problem

$$\varepsilon y'' - \frac{1}{1+2x} y' + y = 0, \qquad \varepsilon \ll 1, \qquad y(0) = 1, \qquad y(1) = 0$$

As always, remember to check and explain the location of any boundary layer(s).

6. [9 points] Short answer questions.

(a) Sketch a bifurcation diagram, with respect to the parameter h, for the autonomous ODE $u' = u^2 - h$. Label your axes, and which parts are stable or unstable.

(b) Write a little-o relation stating that $\log \varepsilon$ blows up more weakly than any negative power of ε, as $\varepsilon \to 0^+$, then prove it.

(c) Is $f(\lambda, t) = 1/t^\lambda$ pointwise, and/or uniformly, convergent to zero on the interval $t \in (1, \infty)$, as $\lambda \to +\infty$? (briefly explain)

Useful formulae:

non-oscillatory WKB approximation

$$y = \frac{1}{\sqrt{k(x)}} e^{\pm \frac{1}{\varepsilon} \int k(x) dx}$$

Binomial

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \cdots$$

Error function [note $\mathrm{erf}(0) = 0$ and $\lim_{z\to\infty} \mathrm{erf}(z) = 1$]:

$$\mathrm{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2} ds$$

Euler relations

$$e^{i\theta} = \cos\theta + i\sin\theta, \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Power-reduction identities

$$\begin{aligned} \cos^3\theta &= \frac{1}{4}(3\cos\theta + \cos 3\theta) \\ \cos^2\theta \sin\theta &= \frac{1}{4}(\sin\theta + \sin 3\theta) \\ \cos\theta \sin^2\theta &= \frac{1}{4}(\cos\theta - \cos 3\theta) \\ \sin^3\theta &= \frac{1}{4}(3\sin\theta - \sin 3\theta) \end{aligned}$$