Math 46, Applied Math (Spring 2009): Midterm 2

2 hours, 50 points total, 7 questions, varying numbers of points (also indicated by space)

1. [5 points] Find a 2-term asymptotic expansion for $I(\lambda) = \int_\lambda^\infty t^2 e^{-t^2} dt$ in the large positive parameter $\lambda \to \infty$.

2. [7 points]

(a) Write the first 3 terms (i.e. trivial term plus two more) in the Neumann series for the solution of $u(t) = t + \lambda(Ku)(t)$, where K is a Volterra operator with kernel $k(t, s) = st$ and $\lambda \in \mathbb{R}$ some constant.

(b) Use the fact that this series is always uniformly convergent on any bounded interval to prove that K (acting on any bounded interval) has *no* eigenvalues.

3. [6 points] Use an energy argument to show that the eigenvalues of the following Neumann boundary condition problem have definite sign (which?):

$$\begin{aligned} ((1+x)u')' &= \lambda u \qquad \text{for } 0 < x < 1 \\ u'(0) &= u'(1) \;=\; 0 \end{aligned}$$

Is $\lambda = 0$ an eigenvalue? If so, what is its eigenspace?

4. [10 points] Consider the integral operator $Ku(x) := \int_0^\pi x \sin y \, u(y) dy$ acting on functions on $(0, \pi)$.

(a) What are all eigenvalue(s) (with multiplicity) and eigenspace(s) of this operator?

(b) Give the general solution to $Ku(x) - 2\pi u(x) = \sin x$, or explain why not possible.

(c) Give the general solution to $Ku(x) - \pi u(x) = \sin 2x$, or explain why not possible.

(d) Give the general solution to $Ku(x) = \sin x$, or explain why not possible.

5. [6 points] Consider the integral operator $Ku(x) = \int_1^e k(x,y)u(y)dy$ with kernel

$$k(x,y) = \begin{cases} 1 - \ln y, & x < y \\ 1 - \ln x, & x > y \end{cases}$$

Convert the eigenvalue problem $Ku = \lambda u$ into a Sturm-Liouville problem on the interval $(1, e)$. Don't forget to find homogeneous boundary conditions [Hint: one will be Dirichlet, one Neumann]

6. [8 points] Short-answer questions

(a) Is the sequence of function $\{x, x^2, x^3, \ldots\}$ convergent to 0 in the L^2 sense on the interval $[0,1]$? (prove via a quick calculation)

(b) A symmetric Fredholm integral operator K on (a,b) has eigenvalues $1/n^2$ and normalized eigenfunctions ϕ_n, $n = 1, 2, \ldots$ What condition on f makes the equation $Ku - \frac{1}{4}u = f$ soluble, and what then is the general solution?

(c) On the interval $(0,\pi)$, the functions $\phi_n(x) = \sin nx$ for $n = 1, 2, \ldots$ form an orthogonal set. If $f = \sum_{n=1}^{\infty} f_n \phi_n$ then use the Cauchy-Schwarz inequality to bound f_1 in terms of $\|f\|$. (BONUS: Show that this either stronger or weaker than Bessel's inequality)

(d) [BONUS] Answer question a) for two other forms of convergence on the same interval

7. [8 points] Now some fun new territory! Consider the inhomogenous SLP

$$-u'' = f$$

where f is some given driving function on $[0,1]$, with boundary conditions $u(0) = u(1) = 0$ [Hint: apply them wherever possible below]

(a) Attack the SLP as you would to convert an IVP into a Volterra integral equation, i.e. integrate from 0 to x, twice, and convert any iterated integrals into single ones. Write your result as $u(x) =$ something involving f and the unknown value $u'(0)$.

(b) Now find an expression for $u'(0)$ purely in terms of f, as follows: Multiply both sides of the original ODE by $(x-1)$ then integrate from 0 to 1.

(c) Substitute your expression for $u'(0)$ into part a) to get an explicit solution formula for $u(x)$. FUN BONUS: Show that this is actually equivalent to the familiar Greens function solution (from worksheet) to this SLP!

Useful formulae:

non-oscillatory WKB approximation

$$y = \frac{1}{\sqrt{k(x)}} e^{\pm \frac{1}{\varepsilon} \int k(x) dx}$$

Binomial

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \cdots$$

Error function [note $\mathrm{erf}(0) = 0$ and $\lim_{z \to \infty} \mathrm{erf}(z) = 1$]:

$$\mathrm{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2} ds$$

Euler relations

$$e^{i\theta} = \cos\theta + i \sin\theta, \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Power-reduction identities

$$\begin{aligned}
\cos^3\theta &= \frac{1}{4}(3\cos\theta + \cos 3\theta) \\
\cos^2\theta \sin\theta &= \frac{1}{4}(\sin\theta + \sin 3\theta) \\
\cos\theta \sin^2\theta &= \frac{1}{4}(\cos\theta - \cos 3\theta) \\
\sin^3\theta &= \frac{1}{4}(3\sin\theta - \sin 3\theta)
\end{aligned}$$

Leibniz's formula

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(x,t) dt \;=\; \int_{a(x)}^{b(x)} \frac{df}{dx}(x,t) dt \;-\; a'(x) f(x, a(x)) \;+\; b'(x) f(x, b(x))$$