Math 46, Applied Math (Spring 2008): Midterm 2

2 hours, 50 points total, 6 questions, varying numbers of points (also indicated by space)

1. [5 points]

   Find a 2-term asymptotic expansion for $I(\lambda) = \int_{\lambda}^{\infty} \frac{e^{-t^2}}{t} dt$ in the large positive parameter $\lambda \to \infty$.

2. [8 points]

(a) Write out the first 3 terms (that includes the 'trivial' term) of the Neumann series for the solution to

$$u(t) - \lambda \int_0^t e^{t-s} u(s) ds = e^{-2t}$$

where $\lambda \in \mathbb{R}$ is some constant.

(b) For what values of λ does the full series converge to a unique solution?

3. [10 points] Consider the Sturm-Liouville operator $Au := -u'' - \frac{1}{4}u$ on $[0, \pi]$ with Neumann boundary conditions $u'(0) = u'(\pi) = 0$.

   (a) Find the set of eigenfunctions and corresponding eigenvalues of A.

   (b) Does the equation $Au = f$ with the above boundary conditions have a Green's function? If so, find an expression for it; if not, explain in detail why not.

   (c) Use the Green's function, or if not possible, another ODE solution method, to write an explicit formula for the solution $u(x)$ to $Au = f$ with the above boundary conditions, in terms of a general driving $f(x)$.

(d) [BONUS] What is the spectrum of the Green's operator $Gu(x) = \int_0^\pi g(x,\xi)u(\xi)d\xi$, or the solution operator you used above?

4. [8 points] Consider the set of two functions $\{1, x\}$ on the interval $x \in [0,1]$.

(a) Replace the second function by another one in $\text{Span}\{1, x\}$ which turns the pair into an *orthogonal set.*

(b) Find the best approximation (in the mean-square or L^2 sense) to the function $\ln x$ on $(0,1)$ using this orthogonal set. Don't bother to evaluate integrals; just write expressions for the coefficients. (Note that the function is unbounded but still in $L^2(0,1)$.)

5. [9 points] Consider the integral operator $Ku(x) := \int_0^1 x^3 y u(y) dy$

(a) What are the eigenvalue(s) (with multiplicity) and eigenfunction(s) of this operator?

(b) Give the general solution to $Ku(x) - \frac{1}{10}u(x) = x$, or explain why not possible.

(c) Give the general solution to $Ku(x) - \frac{1}{5}u(x) = x$, or explain why not possible.

(d) [BONUS]: Give the general solution to $Ku(x) = 2x^3$, or explain why not possible.

6. [10 points]

(a) By converting to a Sturm-Liouville problem, find the eigenvalues and eigenfunctions of the operator $Ku(x) := \int_0^1 k(x,y)u(y)dy$ with kernel

$$k(x,y) = \begin{cases} x, & x < y \\ y, & x > y \end{cases}$$

[Hint: you'll need boundary conditions; look for both Dirichlet and Neumann type]

(b) If possible, solve $Ku(x) = \sin(\pi x/2)$.

(c) Discuss limitations on reconstructing $u(x)$ from measured data $f(x) = Ku(x)$ which has been polluted by noise (say 0.01) in each of the eigenfunction coefficients.

(d) [BONUS] Solve b) using a *different* method from the one you used, *i.e.* if you did use the eigenbasis, don't, and visa versa.

Useful formulae:

non-oscillatory WKB approximation

$$y = \frac{1}{\sqrt{k(x)}} e^{\pm \frac{1}{\varepsilon} \int k(x) dx}$$

Binomial

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

Error function [note $\text{erf}(0) = 0$ and $\lim_{z\to\infty}\text{erf}(z) = 1$]:

$$\text{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2} ds$$

Euler relations

$$e^{i\theta} = \cos\theta + i\sin\theta, \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Power-reduction identities

$$\begin{aligned} \cos^3\theta &= \frac{1}{4}(3\cos\theta + \cos 3\theta) \\ \cos^2\theta \sin\theta &= \frac{1}{4}(\sin\theta + \sin 3\theta) \\ \cos\theta \sin^2\theta &= \frac{1}{4}(\cos\theta - \cos 3\theta) \\ \sin^3\theta &= \frac{1}{4}(3\sin\theta - \sin 3\theta) \end{aligned}$$

Leibniz's formula

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = \int_{a(x)}^{b(x)} \frac{df}{dx}(x,t)dt - a'(x)f(x,a(x)) + b'(x)f(x,b(x))$$