Math 46, Applied Math (Spring 2008): Final

3 hours, 80 points total, 9 questions, roughly in syllabus order (apart from short answers)

1. [16 points. Note part c, worth 7 points, is independent of the others]
   A nonlinear damped oscillator is given by the initial-value problem
   $$my'' + ay' + ky^3 = 0 \qquad y(0) = 0 \qquad my'(0) = I$$
   (a) If m is a mass, find the dimensions of the other three parameters a, k, I (recall y is a displacement, *i.e.* length).

   (b) Write down *two* length scales and *two* time scales.

   (c) Show that when the model is non-dimensionalized using scaling appropriate for the *small mass* limit (choose time and length scales which don't involve m), the IVP
   $$\varepsilon y'' + y' + y^3 = 0 \qquad y(0) = 0 \qquad \varepsilon y'(0) = 1$$
   results. What is ε in terms of the original parameters?

(d) Find a *leading-order* perturbation approximation to the solution of the IVP from (b), and give a crude sketch showing any key features. Here it is written out again:

$$\varepsilon y'' + y' + y^3 = 0 \qquad y(0) = 0 \qquad \varepsilon y'(0) = 1 \qquad \varepsilon \ll 1$$

Don't forget to sketch; you can do intuitively even without solving (1 point):

2. [6 points] Formulate the IVP

$$u'' + u' + tu = 1, \quad u(0) = 2, \quad u'(0) = 1$$

as a Volterra integral equation of the form $Ku - \lambda u = f$ (do not try to solve).

3. [8 points] K is a symmetric Fredholm operator on $[0, \pi]$ with continuous kernel, a complete set of (unnormalized) eigenfunctions $\{\sin nx\}$ labeled by $n = 1, 2, \ldots$, with corresponding eigenvalues $1/n^2$.

   (a) Use this to find the general solution to $Ku(x) - 2u(x) = \sin 2x$, or explain why not possible.

   (b) Find the general solution to $Ku(x) - \frac{1}{4}u(x) = \sin 2x$, or explain why not possible.

   (c) Find the general solution to $Ku(x) - \frac{1}{4}u(x) = 1$, or explain why not possible.

4. [9 points] The following PDE describes a chemical with concentration $u(\mathbf{x}, t)$ diffusing while being broken down by the environment at given rate $\alpha(\mathbf{x}) \geq 0$. Prove that any solution to the IVP is unique.

$$u_t = \Delta u - \alpha u \quad \text{in } \Omega, \qquad\qquad u = 0 \quad \text{on } \partial\Omega, \qquad\qquad u(\mathbf{x}, 0) = f(\mathbf{x}) \quad \text{in } \Omega$$

An extra drift term is added to the right-hand side of the PDE giving $u_t = \Delta u - \mathbf{c} \cdot \nabla u - \alpha u$. Find a condition on the drift velocity field $\mathbf{c}(\mathbf{x})$ such that your above proof method still works.

5. [5 points] Compute directly the convolution $(u * v)(x)$ of the following two functions on $\mathbb{R}$:

$$u(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases} \qquad \text{(this is the unit step function),}$$

$$v(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \qquad \text{(this is a top-hat function).}$$

6. [7 points] Use Fourier transforms to compute the convolution of the Cauchy distribution function

$$u(x) = \frac{1}{1 + x^2}$$

with itself.

[BONUS: describe precisely the linear transformation required to take the original Cauchy function to your above answer, and interpret].

7. [9 points] Use Fourier transforms to solve the 1D wave equation $u_{tt} = u_{xx}$ for $x \in \mathbb{R}$, $t > 0$, with initial conditions $u(x, 0) = 0$, and $u_t(x, 0) = f(x)$ for a general function f. Try to give an answer involving a real-space integral. [Hint: after you use the ICs, combine things to make a trig function]

[BONUS: describe in words the action that propagation in time has upon the initial function, and make a connection to image processing].

8. [10 points] Short answers.

(a) Is the PDE $u_{xx} + u_{yy} = 4u_{xy}$ parabolic, hyperbolic or elliptic?

(b) Find the *general* solution to the PDE $u_{xy} = 1$ for $x, y \in \mathbb{R}$.

(c) The speed c of sound in a gas depends only on density ρ and pressure P (dimensions $ML^{-1}T^{-2}$). Deduce as much as you can about their relationship.

(d) Use the Cauchy-Schwarz inequality to give an upper bound to the number $\int_0^1 y^2 f(y)dy$ in terms of $\|f\|$ on the interval $(0, 1)$.

(e) [BONUS] The 2-norm of an operator is defined as $\max_{f \neq 0} \|Kf\|/\|f\|$. Compute the 2-norm of the Fredholm operator with kernel xy on the interval $[0, 1]$.

9. [10 points] More short answers!

(a) Sketch a bifurcation diagram, including stability, for the autonomous ODE $u' = u^2 - h$.

(b) In the limit $n \to +\infty$ does the top-hat sequence $f_n(x) = n^{-1/2}$ for $x < n$, zero otherwise, converge to the zero function on $[0, \infty)$ pointwise? uniformly? in L^2 sense? (three binary answers required)

(c) Define *completeness* for a set of functions $\{\phi_j\}_{j=1,2,\ldots}$ on an interval $[a, b]$.

(d) The *auto-correlation* of a (complex-valued) function $u(x)$ is defined as $C(x) = \int_{-\infty}^{\infty} \overline{u(y-x)}u(y)dy$ (note there is no typo, and bar means complex conjugate), and is useful in signal processing. Find its Fourier transform $\hat{C}(\xi)$ in terms of $\hat{u}(\xi)$. (This is called the Wiener-Khintchine theorem).

Useful formulae

Non-oscillatory WKB approximation

$$y = \frac{1}{\sqrt{k(x)}} e^{\pm \frac{1}{\varepsilon} \int k(x) dx}$$

Binomial

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

Error function [note $\text{erf}(0) = 0$ and $\lim_{z\to\infty}\text{erf}(z) = 1$]:

$$\text{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2} ds$$

Euler relations

$$e^{i\theta} = \cos\theta + i\sin\theta, \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Power-reduction identities

$$\begin{aligned}
\cos^3\theta &= \frac{1}{4}(3\cos\theta + \cos 3\theta) \\
\cos^2\theta \sin\theta &= \frac{1}{4}(\sin\theta + \sin 3\theta) \\
\cos\theta \sin^2\theta &= \frac{1}{4}(\cos\theta - \cos 3\theta) \\
\sin^3\theta &= \frac{1}{4}(3\sin\theta - \sin 3\theta)
\end{aligned}$$

Leibniz's formula

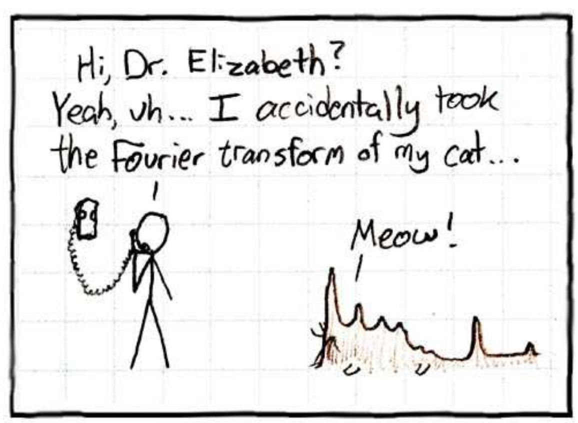
$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = \int_{a(x)}^{b(x)} \frac{df}{dx}(x,t)dt - a'(x)f(x,a(x)) + b'(x)f(x,b(x))$$

Fourier Transforms: $\hat{u}(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} u(x)dx$

$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}\hat{u}(\xi)d\xi$

$u(x)$	$\hat{u}(\xi)$
$\delta(x-a)$	$e^{ia\xi}$
e^{ikx}	$2\pi\delta(k+\xi)$
e^{-ax^2}	$\sqrt{\frac{\pi}{a}} e^{-\xi^2/4a}$
$e^{-a\lvert x\rvert}$	$\frac{2a}{a^2+\xi^2}$
$H(a - \lvert x\rvert)$	$2\frac{\sin(a\xi)}{\xi}$
$u^{(n)}(x)$	$(-i\xi)^n\hat{u}(\xi)$
$u * v$	$\hat{u}(\xi)\hat{v}(\xi)$

Here $H(x) = 1$ for $x \geq 0$, zero otherwise.



Greens first identity: $\int_\Omega u\Delta v + \nabla u \cdot \nabla v \, d\mathbf{x} = \int_{\partial\Omega} u\frac{\partial v}{\partial n} dA$

Product rule for divergence: $\nabla \cdot (u\mathbf{J}) = u\nabla \cdot \mathbf{J} + \mathbf{J} \cdot \nabla u$