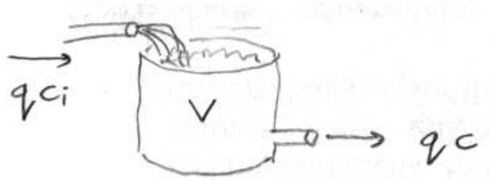
MATH 46 WORKSHEET : Scaling & Non-Dimensionalizing F 3/30/07

Consider a chemical reactor tank with flow rate q, volume V, incoming concentration of reactant c_i. We stir the tank so concentration inside, $c(t)$, is uniform, so (chemical) mass inside is $Vc(t)$.



Inside the tank the reactant decays at rate k, ie rate of loss of mass is $k\underbrace{\boxed{Vc(t)}}_{\text{mass inside}}$

Write an ODE expressing mass balance : $\frac{d}{dt}(Vc(t)) = \underbrace{}_{\text{mass arrival rate}} - \underbrace{}_{\text{loss rate}}$

divide it by V:

Initial Condition is $c(0) = c_0$

How many params in model?

- Rewrite ODE & IC using general nondimensionalized $\bar{t} = \frac{t}{t_c}$, $\bar{c} = \frac{c}{c_c}$ (leave t_c, c_c general for now)

A) Choose $t_c = k^{-1}$ and $c_c = c_i$, rewrite ODE & IC, expressing in terms of $\gamma := \frac{c_i}{c_0}$ and $\beta := \frac{kV}{q}$ (dimensionless params).

B) Instead change t_c to the other timescale derivable from original problem params. Keep c_c as before & rewrite ODE & IC

ie which ODE has sensible limit as $\beta \to 0$?

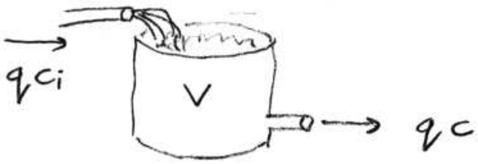
C) if in reality $\beta \ll 1$ (what is the interpretation?) which of A or B is appropriate?

MATH 46 WORKSHEET: Scaling & Non-Dimensionalizing — F 3/30/07

~ SOLUTIONS ~

... also see §1.2.2 in book → it's worked out in full.

Consider a chemical reactor tank with flow rate q, volume V, incoming concentration of reactant c_i. We stir the tank so concentration inside, $c(t)$, is uniform, so (chemical) mass inside is $Vc(t)$.



Inside the tank the reactant <u>decays</u> at rate k, ie rate of loss of mass is $k\boxed{Vc(t)}$ (mass inside)

Write an ODE expressing <u>mass balance</u>:

$$\frac{d}{dt}\big(Vc(t)\big) = \overbrace{qc_i}^{\text{mass arrival rate}} - \overbrace{(qc + kVc)}^{\text{loss rate}}$$

Dimensions of all params:

$$\begin{array}{c|c} & \begin{array}{ccccc} V & q & k & c_i & c_0 \end{array} \\ \begin{array}{c} M \\ L \\ T \end{array} & \begin{bmatrix} & & & 1 & 1 \\ +3 & 3 & & -3 & -3 \\ & -1 & -1 & & \end{bmatrix} \end{array}$$

divide it by V: $\quad \dot{c} = \frac{q}{V}(c_i - c) - kc$

Initial Condition is $c(0) = c_0$

How many params in model? 5: $V \quad q \quad k \quad c_0 \quad c_i$

- Rewrite ODE & IC using general nondimensionalized $\bar{t} = \frac{t}{t_c}$, $\bar{c} = \frac{c}{c_c}$ (leave t_c, c_c general for now).

Use the rules: from lecture
$$\begin{cases} c = c_c \bar{c} \\ \dot{c} = \frac{c_c}{t_c}\frac{d\bar{c}}{d\bar{t}} \end{cases}$$
sub. into ODE & IC

ODE $\quad \frac{c_c}{t_c}\frac{d\bar{c}}{d\bar{t}} = \frac{q}{V}(c_i - c_c\bar{c}) - kc_c\bar{c}$

IC $\quad c_c\bar{c}(0) = c_0$

A) Choose $\boxed{t_c = k^{-1} \text{ and } c_c = c_i}$, rewrite ODE & IC, expressing in terms of $\gamma := \frac{c_i}{c_0}$ and $\beta := \frac{kV}{q}$ (dimensionless params).

sub. these into above:

$$c_i k \frac{d\bar{c}}{d\bar{t}} = \frac{q}{V}(c_i - c_i\bar{c}) - kc_i\bar{c} \quad \overset{\text{cancel } c_i, k}{\Rightarrow} \quad \frac{d\bar{c}}{d\bar{t}} = \frac{1}{\beta}(1 - \bar{c}) - \bar{c}$$

IC $\quad \bar{c}(0) = \frac{1}{\gamma}$

B) Instead change t_c to the $\boxed{\text{other timescale}}$ derivable from original problem params. Keep c_c as before & rewrite ODE & IC → look at dimensions matrix: $\left[\frac{V}{q}\right] = T$ time is the only other timescale.

$$t_c = \frac{V}{q} \Rightarrow c_i\frac{q}{V}\frac{d\bar{c}}{d\bar{t}} = \frac{q}{V}(c_i - c_i\bar{c}) - kc_i\bar{c}$$

$$\Rightarrow \frac{d\bar{c}}{d\bar{t}} = 1 - \bar{c} - \beta\bar{c} \quad \text{with IC } \bar{c}(0) = \frac{1}{\gamma}$$

C) if in reality $\beta \ll 1$ (what is the interpretation?) decay is dominated by flow rate — which of A or $\boxed{B}$ is appropriate? ie which ODE has sensible limit as $\beta \to 0$? B, since in A, β^{-1} diverges → ∞