

MATH 46 WORKSHEET : Converting IVP  $\rightarrow$  Volterra.

9/1/08  
Barnett

using the Lemma :  $\int_0^x \int_0^y u(r) dr dy = \int_0^x (x-y)u(y) dy$

• Convert the IVP : 
$$\begin{cases} u'' + q(t)u = g(t) \\ u(0) = A \\ u'(0) = B \end{cases}$$

into a Volterra equation of the form 
$$\underbrace{Ku}_{\text{integral operator}} - \lambda u = f$$

[Hint: integrate  $\int_0^t ds$  twice:]

• If time, now try on 
$$u'' + p(t)u' + q(t)u = g(t)$$

[see p. 233]

[Hint: use by parts after first  $\int_0^t ds$ ]

SOLUTIONS

using the Lemma :  $\int_0^x \int_0^y u(r) dr dy = \int_0^x (x-y)u(y) dy$

• Convert the IVP : 
$$\begin{cases} u'' + q(t)u = g(t) \\ u(0) = A \\ u'(0) = B \end{cases}$$

into a Volterra equation of the form  $Ku - \lambda u = f$   
 $K$  integral operator.

[Hint: integrate  $\int_0^t ds$  twice:]

$$\int_0^t ds \left( \begin{array}{c} u''(s) + q(s)u(s) = g(s) \\ \downarrow \text{Fund. Thm. Calculus.} \\ u'(t) - B + \int_0^t q(s)u(s) ds = \int_0^t g(s) ds \end{array} \right)$$

$$\int_0^t dr \left( \begin{array}{c} \text{change var to } r \text{ from } t. \\ u(t) - Bt - A + \int_0^t \int_0^r \boxed{q(s)u(s)} ds dr + \int_0^t \int_0^r g(s) ds dr \end{array} \right)$$

use as "u" in Lemma.

$$\lambda = -1 \rightarrow u(t) + \int_0^t \underbrace{(t-s)q(s)}_{\text{kernel}} u(s) ds = \underbrace{\int_0^t (t-s)g(s) ds}_{f(t)} + A + Bt$$

• If time, now try on  $u'' + p(t)u' + q(t)u = g(t)$

[Hint: use by parts after first  $\int_0^t ds$ ]

$$\int_0^t ds \left( \begin{array}{c} \text{just do this term since all others are done above} \\ \int_0^t p(s)u'(s) ds \\ \downarrow \text{by parts} \\ -\int_0^t p'(s)u(s) ds + [p(s)u(s)]_0^t \end{array} \right) \rightarrow p(t)u'(t) - Bp(0)$$

$$\int_0^t dr \left( \begin{array}{c} \text{Lemma} \\ -\int_0^t \int_0^r p'(s)u(s) ds + \int_0^t p(r)u'(r) dr - Bp(0)t \\ \downarrow \text{by parts} \\ -\int_0^t (t-s)p'(s)u(s) ds - \int_0^t p'(s)u(s) ds + p(t)u(t) - p(0)A \end{array} \right)$$

first 3 terms all contribute to kernel.

See p. 233-4.