MATH 46 WORKSHEET : Initial layers

M. 4/14/08

Consider small-mass damped mass-spring IVP:

$$\begin{cases} \varepsilon y'' + y' + y = 0 & t > 0 \\ y(0) = 0 \ , \ \varepsilon y'(0) = 1 & \varepsilon \ll 1 \end{cases}$$

A) Write down & solve for the outer layer : [Hint: $\varepsilon = 0$]

You will find you don't yet know the overall const. (call it A).

B) Rescale the ODE to be in terms of time $\tau = \frac{t}{\delta}$:

C) Use dominant balancing to choose the scale $\delta = \varepsilon^{\alpha}$ for some α [Hint: should agree with lecture]

D) Use this to write rescaled ODE as $Y'' + \cdots + \text{small} = 0$ then set $\varepsilon = 0$ & find general inner layer solution:

E) Match consts in inner layer solution to the ICs.

If time...

F) Use this to get const A from above, write uniform approximation to solution:

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~ SOLUTIONS ~

Consider small-mass damped mass-spring IVP:
$$\begin{cases} \varepsilon y'' + y' + y = 0 & t>0 \\ y(0)=0, \quad \varepsilon y'(0) = 1 & \varepsilon \ll 1 \end{cases}$$

A) Write down & solve for the outer layer : [Hint: $\varepsilon = 0$]

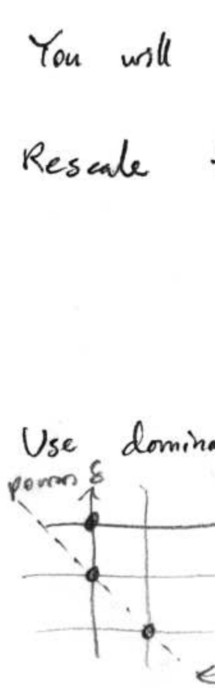
$$y_0' + y_0 = 0 \quad \rightarrow \quad y_0(t) = Ae^{-t}$$

You will find you don't yet know the overall const. (call it A).

B) Rescale the ODE to be in terms of time $\tau = \frac{t}{\delta}$: each $\frac{d}{dt}$ deriv brings in factor of $\frac{1}{\delta}$.

$$\frac{\varepsilon}{\delta^2} Y'' + \frac{1}{\delta} Y' + Y = 0 \qquad \text{with } Y(0)=0, \quad \frac{\varepsilon}{\delta} Y'(0) = 1$$

C) Use dominant balancing to choose the scale $\delta = \varepsilon^{\alpha}$ for some α [Hint: should agree with lecture]



slope $=-1$ $\rightarrow \delta = \varepsilon^1$

$$\frac{\varepsilon}{\varepsilon^2} Y'' + \frac{1}{\varepsilon} Y' + Y = 0$$

mult by ε $\hookrightarrow$ $$Y'' + Y' + \varepsilon Y = 0$$

D) Use this to write rescaled ODE as $Y'' + \cdots + \text{small} = 0$ here.

then set $\varepsilon = 0$ & find general inner layer solution:

$$Y_i'' + Y_i' = 0 \quad \rightarrow \quad Y_i'(\tau) = Be^{-\tau} \quad \rightarrow \quad Y_i(\tau) = Be^{-\tau} + C$$

E) Match consts in inner layer solution to the ICs. are $Y_i(0)=0$, $Y_i'(0) = 1$ from rescaled velocity IC.

$Y_i(0) = 0$ so $C = -B$, $Y_i'(0) = 1$ so $B = -1$ $\rightarrow Y_i(\tau) = 1 - e^{-\tau}$

or $y_i(t) = 1 - e^{-t/\varepsilon}$

If time...

F) Use this to get const A from above, write uniform approximation to solution:

$$\underbrace{\lim_{t\to 0} y_0(t)}_{A} = c_m = \underbrace{\lim_{\tau\to\infty} Y_i(\tau)}_{1-0=1}$$ so $A=1$. Soln. $y_u(t) = y_0 + y_i - c_m = e^{-t} - e^{-t/\varepsilon}$