~ SOLUTIONS ~

Math 46, Applied Math (Spring 2008): Midterm 2

(early version)

2 hours, 50 points total, 6 questions, varying numbers of points (also indicated by space)

was cut from final version.

1. [9 points] → 5 pts in final version.

(a) Formulate the IVP

$$u'' - tu = t, \quad u(0) = 2, \quad u'(0) = 1$$

as a Volterra integral equation of the form $Ku - \lambda u = f$ (do not try to solve).

rewrite using s,

$$u''(s) - s\,u(s) = s \xrightarrow{\int_0^t ds} u'(t) - \underbrace{u'(0)}_{=1} - \int_0^t s\,u(s)\,ds = \frac{t^2}{2}.$$

integrate again

$$u(t) - \underbrace{u(0)}_{=2} - t - \underbrace{\int_0^t \int_0^{s} r\,u(r)\,dr\,ds}_{\int_0^t (t-s)\,s\,u(s)\,ds \text{ via Lemma}} = \frac{t^3}{6}$$

use in lemma.

$$\Rightarrow \int_0^t \underbrace{(t-s)s}_{k(t,s)}\, u(s)\,ds - u(t) = \underbrace{-\frac{t^3}{6} - t - 2}_{f(t) \text{ driving.}}$$

$\lambda = +1$

(b) Find a 2-term asymptotic expansion for $I(\lambda) = \int_\lambda^\infty \frac{e^{-t^2}}{t} dt$ in the large positive parameter $\lambda \to \infty$.

$$I(\lambda) = \int_\lambda^\infty \underbrace{\frac{1}{-2t^2}}_{u} \underbrace{(-2t e^{-t^2})}_{v'}\,dt = \left[\frac{e^{-t^2}}{-2t^2}\right]_\lambda^\infty - \int_\lambda^\infty t^{-3} e^{-t^2}\,dt$$

$$= \frac{e^{-\lambda^2}}{2\lambda^2} - \int_\lambda^\infty \underbrace{\left(-\tfrac{1}{2}t^{-4}\right)}_{u}\underbrace{\left(-2t e^{-t^2}\right)}_{v'}dt$$

$$= \frac{e^{-\lambda^2}}{2\lambda^2} - \left[-\tfrac{1}{2}t^{-4} e^{-t^2}\right]_\lambda^\infty + \underbrace{\int u'v\,dt}_{\text{... gives higher order}}$$

$$= \frac{e^{-\lambda^2}}{2\lambda^2} - \frac{e^{-\lambda^2}}{2\lambda^4} + \cdots$$

2. [7 points]

Volterra kernel

(a) Write out the first 3 terms (that includes the 'trivial' term) of the Neumann series for the solution to

$$u(t) - \lambda \int_0^t e^{t-s} u(s) ds = e^{-2t}$$

where $\lambda \in \mathbb{R}$ is some constant.

$\leftarrow \; u - \lambda K u = f$

ie $(1 - \lambda K) u = f$

$$\Rightarrow \quad u = (1 - \lambda K)^{-1} f$$

$$= (1 + \lambda K + \lambda^2 K^2 + \cdots) f$$

$$= \underbrace{f}_{e^{-2t}} + \lambda K f + \lambda^2 K^2 f$$

$$(Kf)(t) = \int_0^t e^{t-s} \overbrace{e^{-2s}}^{f(s)} ds = e^t \int_0^t e^{-3s} ds = e^t \left[\frac{e^{-3s}}{-3}\right]_0^t$$

$$= -\frac{1}{3} e^t (e^{-3t} - 1)$$

write in terms of s

$$(K^2 f)(t) = K(Kf)(t) = \int_0^t e^{t-s} \left(-\frac{1}{3} e^{-2s} + \frac{1}{3} e^s\right) ds$$

$$= -\frac{e^t}{3} \int_0^t (e^{-3s} - 1) ds = \frac{e^t}{3} \left[\frac{e^{-3s}}{+3} + s\right]_0^t = -\frac{e^t}{3} \left[\frac{e^{-3t}}{-3} - t + \frac{1}{3}\right]$$

So $$u(t) = e^{-2t} + \frac{\lambda}{3}(e^t - e^{-2t}) + \frac{\lambda^2}{3}\left(\frac{e^{-2t}}{3} + te^t - \frac{e^t}{3}\right) + \cdots$$

(b) For what values of λ does the full series converge to a unique solution?

for all λ, (since Picard's method shows terms have $\frac{1}{n!}$ in them, which beat M^n for any M)

see class notes on Picard.

3. [10 points] Consider the Sturm-Liouville operator $Au := -u'' - \frac{1}{4}u$ on $[0, \pi]$ with Neumann boundary conditions $u'(0) = u'(\pi) = 0$.

defined by $Au = \lambda u$.

(a) Find the set of eigenfunctions and corresponding eigenvalues of A.

$$-u'' - \frac{1}{4}u = \lambda u \quad \Rightarrow \quad u(x) = C\cos\sqrt{\tfrac{1}{4}+\lambda}\,x + B\sin\sqrt{\tfrac{1}{4}+\lambda}\,x$$

char eqn. $r^2 + (\frac{1}{4}+\lambda) = 0$

$B=0$ since $u'(0)=0$

$u'(\pi)=0$ $\quad \sin\left(\sqrt{\tfrac{1}{4}+\lambda}\,\pi\right) = 0$ $\quad$ ie $\sqrt{\tfrac{1}{4}+\lambda}\,\pi = n\pi$ $\quad$ ie $\lambda_n = n^2 - \frac{1}{4}$

for $n = 0, 1, 2 \ldots$ $\qquad n = 0, 1, 2, \ldots$

eigenfunctions $u_n(x) = \cos\left(\sqrt{\tfrac{1}{4}+\lambda_n}\,x\right)$

$= \cos nx$

(b) Does the equation $Au = f$ with the above boundary conditions have a Green's function? If so, find an expression for it; if not, explain in detail why not.

since $\lambda = 0$ not an eigenvalue, does have a Green's function.

solve u_1, u_2 : $\quad Au_1 = 0$ w/ $u_1'(0) = 0$ $\quad$ so $u_1(x) = \cos\frac{x}{2}$

$Au_2 = 0$ w/ $u_2'(\pi) = 0$ $\quad$ so $u_2(x) = C\cos\frac{x}{2} + B\sin\frac{x}{2}$

this forces $C=0$ so $u_2(x) = \sin\frac{x}{2}$.

$$W[u_1,u_2] = u_1u_2' - u_1'u_2 = \tfrac{1}{2}\cos\tfrac{x}{2}\cos\tfrac{x}{2} - \left(-\tfrac{1}{2}\right)\sin\tfrac{x}{2}\sin\tfrac{x}{2} = \tfrac{1}{2}.$$

$$g(x,\xi) = -\frac{1}{p(\xi)W(\xi)}\begin{cases} u_1(x)\,u_2(\xi) & x<\xi \\ u_2(x)\,u_1(\xi) & x>\xi \end{cases} = -2\begin{cases} \cos\frac{x}{2}\sin\frac{\xi}{2} & x<\xi \\ \sin\frac{x}{2}\cos\frac{\xi}{2} & x>\xi. \end{cases}$$

here $p(\xi) = 1$ from SLP.

(c) Use the Green's function, or if not possible, another ODE solution method, to write an explicit formula for the solution $u(x)$ to $Au = f$ with the above boundary conditions, in terms of a general driving $f(x)$.

$$u(x) = \int_0^\pi g(x,\xi)f(\xi)\,d\xi = -2\int_0^x \sin\tfrac{x}{2}\cos\tfrac{\xi}{2}\,f(\xi)\,d\xi - 2\int_x^\pi \cos\tfrac{x}{2}\sin\tfrac{\xi}{2}\,f(\xi)\,d\xi$$

(d) [BONUS] What is the spectrum of the Green's operator $Gu(x) = \int_0^\pi g(x,\xi)u(\xi)d\xi$, or the solution operator you used above?

$G = L^{-1}$ so spectrum is set of $\frac{1}{\lambda_n}$ ie $\frac{1}{n^2 - 1/4}$, $n = 0, 1, \ldots$

4. [7 points] Consider the set of two functions $\{1, x\}$ on the interval $x \in [0, 1]$.

(a) Replace the second function by another one in Span$\{1, x\}$ which turns the pair into an *orthogonal set.*

$$f_1(x) = 1$$

$$(f_1, x) = \int_0^1 1 \cdot x \, dx = 1/2$$

so $f_2(x) = x - \frac{(f_1, x)}{\|f_1\|^2} = x - \frac{1/2}{1} = x - 1/2$

$\{1, x - 1/2\}$ are orthogonal (not orthonormal)

(b) Find the best approximation (in the mean-square or L^2 sense) to the function $\ln x$ on $(0, 1)$ using this orthogonal set. (Note that the function is unbounded but still in $L^2(0, 1)$.)

coeffs $c_i = \frac{(f, f_i)}{\|f_i\|^2}$ give best approximation $\sum_{i=1}^{2} c_i f_i$ to f.

$$c_1 = \frac{(f, f_1)}{\|f_1\|^2} = \frac{(\ln x, 1)}{\int_0^1 1^2 dx} = \int_0^1 \ln x \, dx = \left[x \ln x\right]_0^1 - \int_0^1 x \cdot \frac{1}{x} dx$$

$$= -1$$

$$c_2 = \frac{(f, f_2)}{\|f_2\|^2} = \frac{(\ln x, x - 1/2)}{\int_0^1 (x - 1/2)^2 dx} = \frac{\int_0^1 x \ln x \, dx - \frac{1}{2}\int_0^1 \ln x \, dx}{2 \int_0^{1/2} x^2 dx}$$

$$= \frac{\left[\frac{x^2}{2} \ln x\right]_0^1 - \int_0^1 \frac{x^2}{2} \cdot \frac{1}{x} dx + \frac{1}{2}}{2 \cdot \frac{1}{3} \cdot \frac{1}{8}}$$

$$= \frac{1/4}{1/12} = 3.$$

$$f(x) \approx -1 + 3(x - 1/2)$$

$\alpha_1(x) = x^3$ $\qquad \beta_1(x) = x$

5. [8 points] Consider the integral operator $Ku(x) := \int_0^1 x^3 y u(y) dy$

(a) What are the eigenvalue(s) (with multiplicity) and eigenfunction(s) of this operator?

$$A = [(\alpha_1, \beta_1)] = \left[\int_0^1 x^3 \cdot x \, dx\right] = [1/5]$$

so $\lambda = 1/5$ eigenvalue (multiplicity 1) w/ efunc $\alpha_1(x) = x^3$

Also $\lambda = 0$ ∞-multiplicity eigenvalue w/ eigenspace $\text{Span}\{x\}^\perp$

ie all funcs. orthog to the func. x on $[0,1]$.

(b) Give the general solution to $Ku(x) - \frac{1}{10}u(x) = x$, or explain why not possible.

$\lambda = \frac{1}{10} \neq$ eigenvalue $\Rightarrow$ unique solution exists.

taking inner prod w/ β_j gives lin. alg:

$A\vec{c} - \lambda\vec{c} = \vec{f}$ ie $\frac{1}{5}c - \frac{1}{10}c = \frac{1}{3}$ ie $c = \frac{10}{3}$

$f_1 = (x, \beta_1) = \int_0^1 x \cdot x \, dx = 1/3$

(*) then $\sum c_j \alpha_j(x) - \lambda u(x) = f(x)$

so $u(x) = 10\left(\frac{10}{3}x^3 - x\right)$

(c) Give the general solution to $Ku(x) - \frac{1}{5}u(x) = x$, or explain why not possible.

$\lambda = \frac{1}{5}$ is eigenvalue.

$\Rightarrow$ no solution unless $f_1 = 0$, which it isn't $\Rightarrow$ no solution.

+2 (d) [BONUS]: Give the general solution to $Ku(x) = \underbrace{2x^3}_{f(x)}$, or explain why not possible.

K has a zero eigenvalue, so either no solution or ∞ number of them.

$f(x)$ is in $\mathrm{Span}\{\alpha_j(x)\}$ ie $\mathrm{Ran}\, K$, so there is a solution.

Find one solution: $\int_0^1 x^3 y\, u(y)\,dy = 2x^3$ ie $\int_0^1 y\, u(y)\, dy = 2$

ie $u = 4 \ \forall x$ works. $\Rightarrow$ Gen soln. $u(x) = 4 + \underbrace{(\text{any func} \perp \text{to } x)}_{\mathrm{Nul}\, K}$.

6. [9 points]

(a) By converting to a Sturm-Liouville problem, find the eigenvalues and eigenfunctions of the operator $Ku(x) := \int_0^1 k(x,y)u(y)dy$ with kernel

$$k(x,y) = \begin{cases} x, & x < y \\ y, & x > y \end{cases}$$

if efunc: [Hint: you'll need boundary conditions; look for both Dirichlet and Neumann type]

$$\lambda u(x) = (Ku)(x) = \int_0^x y\, u(y)\, dy + \int_x^1 x\, u(y)\, dy \qquad (*)$$

Leibniz

$$\text{so } \lambda u'(x) = \cancel{x u(x)} + \int_x^1 u(y)\, dy - \cancel{x u(x)} = \int_x^1 u(y)\,dy \qquad (\dagger)$$

$$\text{so } \lambda u''(x) = -u(x)$$ Leibniz again

BCs: $u(0) = 0$ from looking at $(*)$

but cannot deduce anything about $u(1)$.

Rather use $(\dagger)$ to deduce $u'(1) = 0$

$\Rightarrow$ SLP $\quad u'' + \frac{1}{\lambda}u = 0 \qquad u(0) = 0, \quad u'(1) = 0$ mixed type BCs.

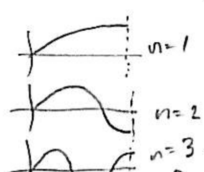
gen. soln. ($\lambda > 0$) $\quad A\cos\frac{1}{\sqrt{\lambda}}x + B\sin\frac{1}{\sqrt{\lambda}}x$

Left BC gives $A = 0$

Right BC gives $\cos\frac{1}{\sqrt{\lambda}} = 0$

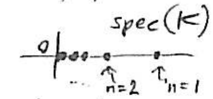
ie $\frac{1}{\sqrt{\lambda}} = (n - \frac{1}{2})\pi, \quad n = 1, 2, \cdots$

$\lambda_n = \frac{1}{\pi^2 (n-\frac{1}{2})^2}$



note: modes of open-closed pipe in acoustics!

spec(K)



efuncs are $\phi_n(x) = \sin\frac{x}{\sqrt{\lambda_n}} = \sin[(n-\frac{1}{2})\pi x]$ (unnormalized)

etc.

(b) If possible, solve $Ku(x) = \sin(\pi x/2)$. this is an eigenfunction $(n=1)$

Using eigenbasis: so $f(x) = \sum_{i=1}^{\infty} f_i \phi_i(x)$ is just $f_1 = 1$, $f_n = 0$, $n>1$

use $c_i = \dfrac{f_i}{\lambda_i - 0}$ $\forall i$ leaves just one term $(i=1)$ $c_1 = \dfrac{f_1}{\lambda_1} = \dfrac{\pi^2}{4}$

So $u(x) = \sum c_i \phi_i(x) = \dfrac{\pi^2}{4} \sin\dfrac{\pi x}{2}$ unique.

(c) Discuss limitations on reconstructing $u(x)$ from measured data $f(x) = Ku(x)$ which has been polluted by noise (say ~~1%~~ (0.01)) in each of the eigenfunction coefficients.

This is not a convolution kernel, but same ideas apply : K is simply multiplication by λ_n in the eigenfunc. basis $\{\phi_j\}$. ie $f_i = \lambda_i c_i$

$\Rightarrow$ to reconstruct. $u(x)$ from noisy meas. data $Ku(x)$ we get $c_i = \dfrac{f_i}{\lambda_i}$ (as above).

If noise on $f_i = 0.01$ then all coeffs with $\dfrac{1}{\lambda_i} < 100$ can be reconstructed with error less than roughly 1.

solving gives $(n-\frac{1}{2})^2 = \dfrac{100}{\pi^2}$

ie $n \leq \dfrac{10}{\pi} + \dfrac{1}{2} \approx 4$

Only ≈ 4 coeffs can be reconstructed meaningfully!

(d) [BONUS] Solve b) using a *different* method from the one you used, *i.e.* if you did use the eigenbasis, don't, and visa versa.

We may apply $\frac{d^2}{dx^2}$ to both sides of $Ku(x) = f(x)$ (via Leibniz as before)

Get: $-u(x) = f''(x)$ in $x \in [0,1]$

this is explicit solution for $u(x)$!

Evaluating $f''(x) = -\left(\frac{\pi}{2}\right)^2 \sin\frac{\pi x}{2}$ we get same as in b).