4/23/08.

Math 46, Applied Math (Spring 2008): Midterm 1

SOLUTIONS

2 hours, 50 points total, ~~6~~ 5 questions worth varying number of points

1. [9 points] In modeling an atomic explosion, G. I. Taylor supposed there was a law relating the fireball radius r to time t after explosion, and the two fixed parameters E energy released (units: mass times speed squared) and ρ the initial air density.

   (a) How many independent dimensionless quantities are there? Give them.

$$\begin{array}{c|c} & \begin{array}{cccc} r & t & E & \rho \end{array} \\ \begin{array}{c} M \\ L \\ T \end{array} & \begin{bmatrix} & & 1 & 1 \\ 1 & & 2 & -3 \\ & 1 & -2 & \end{bmatrix} \end{array}$$

$$\pi_1 = \frac{Et^2}{\rho r^5}$$ is the only indep. dim'less param.

This follows since $\dim \operatorname{Nul} A = n - \operatorname{rank} A$
$= 4 - 3 = 1$

   (b) From this deduce as much as you can about how r must scale with t.

A physical law involving only r, t, E, ρ must be of the form $F(\pi_1) = 0$, ie $\pi_1 = c$ ie $\frac{Et^2}{\rho r^5} = c$.

$$\Rightarrow r = \left(\frac{Et^2}{\rho c}\right)^{1/5} = k\left(\frac{E}{\rho}\right)^{1/5} t^{2/5} \sim \underline{t^{2/5}}$$ when E, ρ const.

   (c) If the law is enlarged to include dependence on an extra fixed parameter a, the acceleration due to gravity, use the Buckingham Pi Theorem to deduce whether with all three parameters fixed, the scaling of r with t must be as before.

5 quantities:

$$\begin{array}{c|c} & \begin{array}{ccccc} r & t & E & \rho & a \end{array} \\ \begin{array}{c} M \\ L \\ T \end{array} & \begin{bmatrix} & & 1 & 1 & \\ 1 & & 2 & -3 & 1 \\ & 1 & -2 & & -2 \end{bmatrix} \end{array}$$

$\dim \operatorname{Nul} A = 5 - 3 = 2$

keep π_1 as above

Choose π_2 so that doesn't involve r or t, <u>if possible</u>, since then the Pi Theorem will tell you $\pi_1 = g(\pi_2) = \text{const}$ (not involving t or r)

This is not possible: any dim'less quant must be a 'lin. combo' (in terms of powers) of π_1 and $\pi_2 = \frac{at^2}{r}$. There is no way to cancel both the powers of r & t simultaneously. So all we have is laws such as

$$\frac{Et^2}{\rho r^5} = g\left(\frac{at^2}{r}\right)$$ which generally do <u>not</u> have the simple $r \sim t^{2/5}$ scaling as before.

2. [8 points] Find a uniform approximate solution to the boundary-value problem

$$\varepsilon y'' - (1-x)^2 y' - y = 0, \qquad y(0) = y(1) = 1$$

interval is $[0, 1]$

where $0 < \varepsilon \ll 1$. [Hint: if you think an integral is difficult, it's not; just substitute].

Boundary layer problem, with relative signs of y'' & y' term suggesting BL @ $x=1$.

Outer layer: $(\varepsilon = 0)$

$$-(1-x)^2 y' - y = 0$$

$$\Rightarrow \int \frac{dy}{y} = -\int \frac{dx}{(x-1)^2}$$

$$\Rightarrow \ln|y| = \frac{1}{x-1} + c$$

$$y_0(x) = c e^{\frac{1}{x-1}}, \text{ try}$$

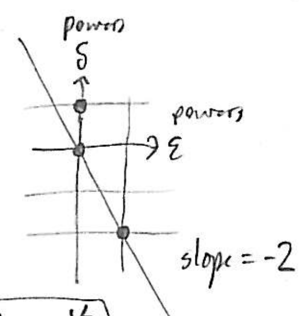
- BL @ $x=0$, match $y_0(1) = 1$ but $e^{-\frac{1}{0}} \to 0$ so $c \to \infty$, not possible.
- BL @ $x=1$, match $y_0(0) = 1$ ie $c = e^1$ $\Rightarrow y_0(x) = e^{1 + \frac{1}{x-1}} = e^{\frac{x}{x-1}}$

BL @ $x=1$: $\quad \xi = \frac{1-x}{\delta}$ rescale so $y \to Y$, $y' \to -\frac{Y'}{\delta}$, $y'' \to \frac{Y''}{\delta^2}$

note

inner ODE: $\quad \underbrace{\frac{\varepsilon}{\delta^2} Y''}\ + \ \delta^2 \xi^2 \frac{Y'}{\delta} \ \underbrace{- \ Y} = 0$ 3 terms:

these two dominant balance. (unusual).



$\Rightarrow \boxed{\delta = \varepsilon^{1/2}}$

sub for δ: $\quad Y'' + \cancel{\varepsilon^{1/2} \xi^2 Y'} - Y = 0$

drop to leading order

$\sim Y'' - Y = 0$ is non-oscillatory, 2nd order

$$\Rightarrow Y_i(\xi) = A e^{-\xi} + \underline{B e^{+\xi}}$$

will not have finite limit as $\xi \to \infty$ unless $B = 0$.

Match BC @ $x=1$ which is $Y_i(0) = 1$, so $A = 1$. $\Rightarrow Y_i(\xi) = e^{-\xi}$

or $y_i(x) = e^{-\frac{1-x}{\sqrt{\varepsilon}}}$

The common limit $c_m = \lim_{\xi \to \infty} Y_i(\xi) = \lim_{x \to 1} y_0(x) = 0$ already agrees (thankfully!)

So $y_u(x) = y_0 + y_i - c_m = e^{\frac{x}{x-1}} + e^{\frac{x-1}{\sqrt{\varepsilon}}}$

This is not a standard Boundary Layer as in the theorem on p. 119, but works fine.

3. [8 points] Consider the linear homogeneous ODE, $-y'' = \lambda(4x - x^2)^2 y$, on $2 < x < 3$.

(a) For what λ is the problem oscillatory, or non-oscillatory, in character?

$(4x - x^2)^2$ always ≥ 0 so $y'' + \lambda k^2(x) y = 0$ is $\begin{cases} \text{oscillatory } \lambda > 0 \\ \text{non-osc. } \lambda < 0 \end{cases}$

(b) Write down an approximate *general* solution to the ODE that is accurate for large *positive* λ.

ie oscillatory case

$$\varepsilon^2 y'' + k^2(x) y = 0 \quad \text{with} \quad \varepsilon^2 = \frac{1}{\lambda}, \; k(x) = 4x - x^2.$$

$$\text{so} \quad y_a(x) = \frac{c_1}{\sqrt{4x-x^2}} \sin\left(\sqrt{\lambda} \int 4x - x^2\, dx\right) + \frac{c_2}{\sqrt{4x-x^2}} \cos\left(\sqrt{\lambda} \int 4x - x^2\, dx\right)$$

Noticing lower end of interval is $x = 2$, we can conveniently choose this as lower limit of definite integral.

ie, eigenvalues.

(c) Use this to get an approximation for the sequence of values λ, *and* corresponding solutions $y(x)$, such that there is a nontrivial solution with boundary conditions $y(2) = 0$ and $y(3) = 0$. [Hint: use the lower boundary condition to make your life easier. Don't forget to write the solutions $y(x)$ too].

eigenfuncs.

at $x=2$, $\int_2^x 4s - s^2\, ds = 0$ so $\sin(\cdot)$ term vanishes, and $\cos(\cdot) = 1$.

$\Rightarrow$ forces $c_2 = 0$.

$$\Rightarrow \quad y_a(x) = \frac{c_1}{\sqrt{4x-x^2}} \sin\left(\sqrt{\lambda} \int_2^x 4s - s^2\, ds\right)$$

integrate $\rightarrow \left[2s^2 - \frac{s^3}{3}\right]_2^x = 2x^2 - \frac{x^3}{3} - \underbrace{\left(8 - \frac{8}{3}\right)}_{\frac{16}{3}}$

Eigenvalue condition $\sqrt{\lambda_n} \int_2^3 k(x)\, dx = n\pi$ ie $\lambda_n = \left(\frac{n\pi}{\int_2^3 k(x)\, dx}\right)^2 = \left(\frac{n\pi}{18 - 9 - \frac{16}{3}}\right)^2 = \left(\frac{3n\pi}{11}\right)^2$

$$\text{Eigenfunctions} \quad y_n(x) = \frac{1}{\sqrt{4x-x^2}} \sin\left[\frac{3n\pi}{11}\left(2x^2 - \frac{x^3}{3} - \frac{16}{3}\right)\right]$$

(d) [BONUS] Find the values λ if the boundary conditions are $y'(2) = 0$ and $y(3) = 0$.

derivative BC so need to compute $y'(2)$ for general solution.... use chain & prod rules:

$$y'(x) = \frac{c_1}{\sqrt{k}} \sqrt{\lambda} k \cos(\cdot) - \frac{c_1}{2k^{3/2}} k' \sin(\cdot) - \frac{c_2}{\sqrt{k}} \sqrt{\lambda} k \sin(\cdot) - \frac{c_2}{2k^{3/2}} k' \cos(\cdot)$$

But, nicely (by design), at $x = 2$, $k'(x) = 4 - 2x$ vanishes!

this means $y'(2)$ is indep. of c_2, so $c_1 = 0$

$$\Rightarrow y(x) = \frac{1}{\sqrt{4x-x^2}} \cos \sqrt{\lambda}\left(2x^2 - \frac{x^3}{3} - \frac{16}{3}\right)$$

and to get $y(3) = 0$ need $\cos(\cdot) = 0$

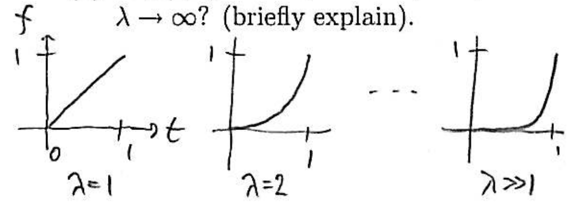
so argument of cos $= (n + \tfrac{1}{2})\pi$.

$$\Rightarrow \lambda_n = \left[\frac{3(n+1/2)\pi}{11}\right]^2 \quad \text{with} \quad y_n(x) = \frac{1}{\sqrt{4x-x^2}} \cos\left[\frac{3(n+\frac{1}{2})\pi}{11}\left(2x^2 - \frac{x^3}{3} - \frac{16}{3}\right)\right]$$

phew!

4. [10 points] Short answer questions.

(a) Is $f(t, \lambda) = t^\lambda$ pointwise, and/or uniformly, convergent to zero on the interval $t \in (0,1)$, as $\lambda \to \infty$? (briefly explain).



etc.

$\forall t \in (0,1)$, $\lim_{\lambda \to \infty} t^\lambda = 0$ ✓

pointwise convergent

But for any $\lambda > 0$, there are always t values approaching 1 such that $t^\lambda \to 1$.

For each λ $\Rightarrow$ $\underbrace{\text{"}\max_{t \in (0,1)}\text{"}}_{\text{officially, "sup"}} t^\lambda = 1$, ie not uniformly convergent.

(b) Does $e^{-t} = o(t^{-\alpha})$ hold as $t \to \infty$, for any fixed $\alpha > 0$? Prove your answer.

Yes it does. We want $\lim_{t \to \infty} \frac{e^{-t}}{t^{-\alpha}} = 0$

Same as $\lim_{t \to \infty} \frac{t^\alpha}{e^t} \xrightarrow{\text{l'Hôpital}} \frac{\alpha t^{\alpha-1}}{e^t} \xrightarrow{\text{l'Hôpital}} \frac{\alpha(\alpha-1)t^{\alpha-2}}{e^t} \cdots \to \frac{C_{n,\alpha} t^{\alpha-n}}{e^t}$

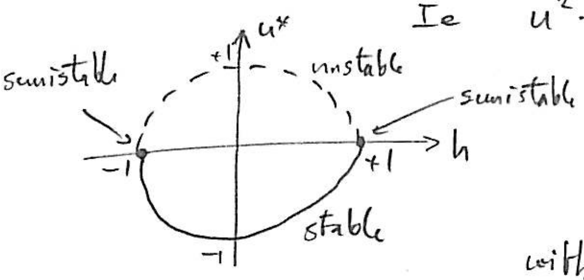
repeat n times where n is smallest integer $\geq \alpha$.

Now $\alpha - n < 0$ so $\lim_{t \to \infty} t^{\alpha - n} \to 0$ and we've proven the claim.

(c) Sketch the bifurcation diagram, in the domain $-1 \leq h \leq 1$, for the autonomous ODE $u' = \underbrace{u^2 + h^2 - 1}_{f(u,h)}$. Label your axes, and which parts are stable or unstable.

Equilibrium where $u' = 0$ ie points in (h,u) plane where $f = 0$.

Ie $u^2 + h^2 - 1 = 0$, ie the unit circle in (h,u) plane.



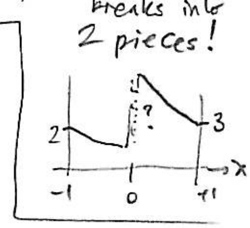
Stability arises from
$f' > 0$ unstable
$f' < 0$ stable.
$f' = 0$ semistable

with $f' = \frac{df}{du} = 2u$.

(d) What, if any, issues do you see in attempting singular perturbation in the problem $\varepsilon y'' + xy' + xy = 0$, $y(-1) = 2$, $y(1) = 3$, for $\varepsilon \ll 1$? (Do not solve the whole thing!)

Interval is $-1 \leq x \leq +1$

left end $(x=-1)$: $\varepsilon y'' + p(x)y' + q(x)y = 0$ has $p(-1) < 0$ so no BL @ $x=-1$
right end: $p(1) > 1$ so no BL @ $x=+1$, either! This means outer layer would have to match both BCs ... not possible (check it's $y_0(x) = ce^{-x}$). Remedy: there's an internal layer! (at $x=0$) where outer layer breaks into 2 pieces!



(e) [BONUS] Modify the interval in part a) so that the uniform convergence property is changed, but not the pointwise convergence.

Any interval $(0, a)$ for $0 < a < 1$ does this (makes both pointwise & uniformly convergent).

If $a \geq 1$, it's not even pointwise.

5. [15 points] Consider the perturbed initial-value problem for $y(t)$ on $t > 0$,

$$y'' + y = 4\varepsilon y (y')^2, \qquad \varepsilon \ll 1, \qquad y(0) = 1, \qquad y'(0) = 0$$

(a) Find a 2-term asymptotic approximation using <u>regular</u> perturbation theory. [Hints: You may find the power-reduction identities on the last page useful. You will get partial credit for leaving the 2^{nd} term as the solution to a clearly-specified IVP.]

$$y = y_0 + \varepsilon y_1 + \cdots$$

Zeroth order: $y_0'' + y_0 = 0$ so $y_0 = A\cos t + B\sin t$

to match ICs, $A = 1$, $B = 0$

$\Rightarrow y_0(t) = \cos t$

Subst. pert. expansion:

$$y_0'' + \varepsilon y_1'' + \cdots + y_0 + \varepsilon y_1 + \cdots = 4\varepsilon\,(y_0 + \varepsilon y_1 \cdots)(y_0' + \varepsilon y_1' + \cdots)^2$$

At $O(\varepsilon^1)$ Terms:

$$\underbrace{y_1'' + y_1}_{\text{homog. solns are } \cos t,\ \sin t} = 4 y_0 y_0'^2 = 4\cos t \sin^2 t = \underbrace{\cos t}_{\text{is homog. soln,}} - \underbrace{\cos 3t}_{\text{is not homog. soln.}}$$

see back page.

So, $y_1(t)$ is solution to the above ODE w/ ICs $y_1(0) = y_1'(0) = 0$.

(from expanding ICs as perturbation series).

Particular solution for driving $= -\cos 3t$ only: Meth. Und. Coeffs.

$y = A\cos 3t$

$\hookrightarrow y'' = -9A\cos 3t$

LHS $= -9Ac + Ac = -c$ = RHS (abbrev for $\cos 3t$)

so $A = 1/8$

Now for driving $= \cos t$, note

$y = Ats$ (abbrev. for $\sin t$.)

$\hookrightarrow y' = Atc + As$

$\hookrightarrow y'' = -Ats + Ac + Ac$

so $y'' + y = \cancel{-Ats} + 2Ac + \cancel{Ats} = 2Ac = c$ = RHS, so $A = \frac{1}{2}$

$$\text{So, } y_1(t) = \overbrace{\tfrac{1}{8}\cos 3t + \tfrac{1}{2} t\sin t}^{\text{particular soln.}} + \underbrace{c_1\cos t + c_2\sin t}_{\text{homog. soln.}}$$

matching ICs gives $c_1 = -1/8$, $c_2 = 0$

$$\text{So } y_a(t) = \cos t + \varepsilon\left(\tfrac{1}{8}(\cos 3t - \cos t) + \underbrace{\tfrac{t\sin t}{2}}_{\text{secular term.}}\right) + O(\varepsilon^2)$$

(b) Is this a uniform approximation for $t \in (0, \infty)$? Why?

No, since the secular term $\varepsilon \frac{t \sin t}{2}$ is unbounded on $t \in (0,\infty)$ for any fixed $\varepsilon > 0$.

(See book: this is standard problem, p. 93)

(c) Use the Poincaré-Lindstedt method to give a more useful 2-term approximation. [Hint: rescale to $\tau = \omega t$ where ω is perturbed from the value 1]

$$\tau = \underbrace{(1 + \varepsilon\omega_1 + \cdots)}_{\omega} t$$

Rescaling just time gives

$$y \to \bar{y}$$
$$y' \to \omega \bar{y}'$$
$$y'' \to \omega^2 \bar{y}''$$

where now prime means $\frac{d}{d\tau}$ (we drop the bars..)

subst:

$$(1+\varepsilon\omega_1 \cdots)^2 (y_0'' + \varepsilon y_1'' \cdots) + y_0 + \varepsilon y_1 + \cdots = 4\varepsilon (y_0 + \cdots)(1+\varepsilon\omega_1 \cdots)^2 (y_0' + \cdots)^2$$

As before, zeroth order gives $y_0(\tau) = \cos\tau$ (since same ICs)

But, at $O(\varepsilon^1)$:

$$2\omega_1 y_0'' + y_1'' + y_1 = 4 y_0 y_0'^2$$

$$\Rightarrow \quad y_1'' + y_1 = \underbrace{4\cos\tau \sin^2\tau}_{\cos\tau - \cos 3\tau} - 2\omega_1 \overbrace{(-\cos\tau)}^{y_0''}$$

so if $\underline{\omega_1 = -\frac{1}{2}}$ this kills any $\cos\tau$ (on-resonance) driving term.

So $y_1(\tau)$ is as before without the $\frac{1}{2} t \sin t$ term, so $c_1 = -1/8$, $c_2 = 0$ as before.

$$\Rightarrow \quad y_1(t) = \frac{1}{8}(\cos 3\tau - \cos\tau)$$

$$y_a(t) = \cos\tau + \frac{\varepsilon}{8}(\cos 3\tau - \cos\tau) + O(\varepsilon^2)$$

where $\tau = (1 - \frac{\varepsilon}{2} + O(\varepsilon^2))t$ is rescaled time. (period got longer a bit).

(d) Is this a uniform approximation for $t \in (0, \infty)$?

yes; that's the point of Poincaré-Linstedt. (but you don't have to prove this).