MATH 46 WORKSHEET : Initial layers M. 4/14/08

Consider small-mass damped mass-spring IVP:
$$\begin{cases} \varepsilon y'' + y' + y = 0 & t>0 \\ y(0) = 0 \ , \quad \varepsilon y'(0) = 1 & \varepsilon \ll 1 \end{cases}$$

A) Write down & solve for the outer layer : [Hint: $\varepsilon = 0$]

You will find you don't yet know the overall const. (call it A).

B) Rescale the ODE to be in terms of time $\tau = \frac{t}{\delta}$:

C) Use dominant balancing to choose the scale $\delta = \varepsilon^{\alpha}$ for some α [Hint: should agree with lecture]

D) Use this to write rescaled ODE as $Y'' + \cdots + \text{small} = 0$ then set $\varepsilon = 0$ & find general inner layer solution:

E) Match consts in inner layer solution to the ICs.

If time...

F) Use this to get const A from above, write uniform approximation to solution:

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~ SOLUTIONS ~

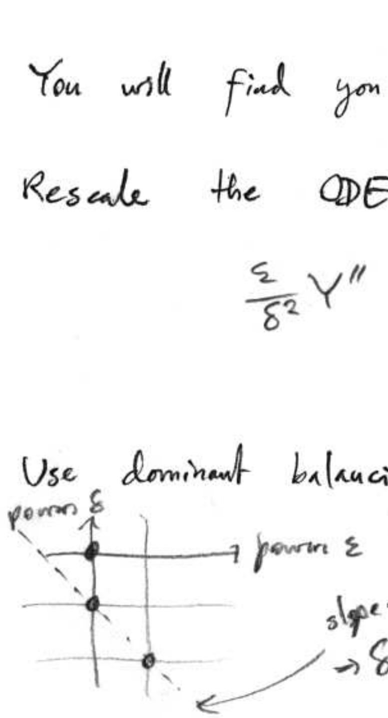
Consider small-mass damped mass-spring IVP:
$$\begin{cases} \varepsilon y'' + y' + y = 0 & t>0 \\ y(0) = 0, \quad \varepsilon y'(0) = 1 & \varepsilon \ll 1 \end{cases}$$

A) Write down & solve for the outer layer : [Hint: $\varepsilon = 0$]
$$y_0' + y_0 = 0 \quad \to \quad y_0(t) = Ae^{-t}$$

You will find you don't yet know the overall const. (call it A).

B) Rescale the ODE to be in terms of time $\tau = \frac{t}{\delta}$: each $\frac{d}{dt}$ deriv brings in factor of $\frac{1}{\delta}$.
$$\frac{\varepsilon}{\delta^2}Y'' + \frac{1}{\delta}Y' + Y = 0 \qquad \text{with } Y(0) = 0, \quad \frac{\varepsilon}{\delta}Y'(0) = 1$$

C) Use dominant balancing to choose the scale $\delta = \varepsilon^\alpha$ for some α [Hint: should agree with lecture]



$$\frac{\varepsilon}{\delta^2}Y'' + \frac{1}{\varepsilon}Y' + Y = 0$$

slope $= -1$ $\to \delta = \varepsilon^1$ mult by ε $\to$ $Y'' + Y' + \varepsilon Y = 0$

here.

D) Use this to write rescaled ODE as $Y'' + \dots + \text{small} = 0$

then set $\varepsilon = 0$ & find general inner layer solution:
$$Y_i'' + Y_i' = 0 \quad \to \quad Y_i'(\tau) = Be^{-\tau} \quad \to \quad Y_i(\tau) = Be^{-\tau} + C$$

E) Match consts in inner layer solution to the ICs. are $Y_i(0) = 0$, $Y_i'(0) = 1$ from rescaled velocity IC.

$Y_i(0) = 0$ so $C = -B$, $Y_i'(0) = 1$ so $B = -1$ $\to Y_i(\tau) = 1 - e^{-\tau}$

or $y_i(t) = 1 - e^{-t/\varepsilon}$

If time...

F) Use this to get const A from above, write uniform approximation to solution:

$$\underbrace{\lim_{t\to 0} y_0(t)}_{A} = c_m = \underbrace{\lim_{\tau\to\infty} Y_i(\tau)}_{1-0=1} \quad \text{so } A=1. \quad \text{Soln. } y_u(t) = y_0 + y_i - c_m = e^{-t} - e^{-t/\varepsilon}$$