5/7/08 · Barnett.

# MATH 46 WORKSHEET : Image (de)convolution in 1d

Consider symm blurring operator $Kf(x) = \int_{-\pi}^{\pi} k(x-y)\, f(y)\, dy$ , $\underset{\uparrow \text{ 'aperture func.'}}{k(s)}$ = even symm, $2\pi$-periodic

A) Show that $\phi_0(x) = 1$ is an eigenfunction of $K$, and find its eigenvalue $\lambda_0$:
[Hint: why is $K\phi_0(x)$ indep. of $x$? why is $\lambda_0$ indep. of $x$?]

B) Show that $\phi_n(x) = \cos nx$, $n=1,2,\cdots$ is eigenfunc. of $K$, find its eigenvalue $\lambda_n$:
[Hint: addition formula, $k$ even]

C) How do $\lambda_n$ relate to Fourier cos coeffs $k_n$ of aperture func $k(s)$?

You could check that $\sin nx$ is also eigenfunc. w/ <u>same</u> eigenval. $\lambda_n$.

Assume image is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos nx + b_n \sin nx \right]$

& after blurring $Kf(x) = g(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[ A_n \cos nx + B_n \sin nx \right]$

D) How are $g$'s Fourier coeffs related to those of $f$?

Such is the nature of convolution kernels. How would you invert $g \to f$, ie deconvolve?

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# MATH 46 WORKSHEET : Image (de)convolution in 1d

## ~ SOLUTIONS ~

Consider symm blurring operator $Kf(x) = \int_{-\pi}^{\pi} k(x-y) f(y)\,dy$, $k(s)$ = even symm $2\pi$-periodic 'aperture func.'

A) Show that $\phi_0(x) = 1$ is an eigenfunction of $K$, and find its eigenvalue $\lambda_0$
[Hint: why is $K\phi_0(x)$ indep. of $x$? why is $\lambda_0$ indep. of $x$?]

$$(K1)(x) = \int_{-\pi}^{\pi} k(x-y)\cdot 1 \cdot dy \underset{\text{change var.}}{\overset{s=y-x}{=}} \int_{-\pi-x}^{\pi-x} k(-s)\,ds \overset{\text{periodicity \& even-symm}}{=} \int_{-\pi}^{\pi} k(s)\,ds.$$ obviously const w.r.t. $x$.

so $\lambda_0 = $ this const $= \int_{-\pi}^{\pi} k(s)\,ds$

B) Show that $\phi_n(x) = \cos nx$, $n=1,2,\cdots$ is eigenfunc. of $K$, find its eigenvalue $\lambda_n$:
[Hint: addition formula, $k$ even]

$$(K\phi_n)(x) = \int_{-\pi}^{\pi} k(x-y)\cos ny\,dy = \int_{-\pi-x}^{+\pi-x} k(-s)\cos n(s+x)\,ds$$

periodic so can shift limit to $(-\pi,\pi]$.

bring out

$$= \int_{-\pi}^{\pi} k(s)\cos ns \cos nx\,ds - \underbrace{\int_{-\pi}^{\pi} k(s)\sin ns \sin nx\,ds}_{\text{zero since } k \text{ even symm}} = \underbrace{\cos nx}_{\phi_n(x)} \cdot \underbrace{\int_{-\pi}^{\pi} k(s)\cos ns\,ds}_{=\lambda_n \text{ eigenval}}$$

bring out

C) How do $\lambda_n$ relate to Fourier cos coeffs $k_n$ of aperture func $k(s)$? $\boxed{\lambda_n = \pi k_n}$

since $k_n = \frac{1}{\pi}\int_{-\pi}^{\pi} k(s)\cos ns\,ds$ Euler-Fourier, or projection, formula.

You could check that $\sin nx$ is also eigenfunc. w/ <u>same</u> eigenval. $\lambda_n$.

Assume image is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$

& after blurring $Kf(x) = g(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}[A_n \cos nx + B_n \sin nx]$

D) How are $g$'s Fourier coeffs related to those of $f$?

Fourier basis = eigenbasis for $K$, so action of $K$ is multiplication in this basis:

$$A_0 = \lambda_0 a_0 = \pi k_0 a_0$$

for $n=1,2,\cdots$
$$A_n = \lambda_n a_n = \pi k_n a_n$$
$$B_n = \lambda_n b_n = \pi k_n b_n$$

so Fourier coeffs get multiplied by $\pi \times$ Fourier coeffs of aperture func.

Such is the nature of convolution kernels. How would you <u>invert</u> $g \to f$, ie deconvolve? <u>divide</u> Fourier coeffs by $\pi k_n$.