Math 46: Applied Math: Homework 9

due Wed May 28—last one!

A) The wave equation in $\mathbb{R}^n$ is $u_{tt} - c^2 \Delta u = 0$, where c is the wave speed.

1. Show that if u is time-harmonic, that is $u(\mathbf{x}, t) = U(\mathbf{x}) \cos \omega t$ for some frequency ω, then U satisfies the Helmholtz equation $(\Delta + E)u = 0$. What is the constant E?
2. Show that the plane wave $U(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}}$, where $\mathbf{k}$ is a constant (wave)vector, is a solution to the Helmholtz equation. [Hint: first hit U with grad, componentwise]. What is the relation between ω and $\mathbf{k}$? (this is the *dispersion relation* for the wave equation).

p.395-398: #4. As a function of ξ this is called a Cauchy distribution. It comes up in statistics and has an infinite variance.

B) Compute the Fourier transform of $g(x) = \sin |x|$. Use this to find a particular solution to the driven mass-spring ODE $u'' + u = f$ in terms of an arbitrary driving function $f(x)$. BONUS: show this differs from the Variation of Parameters solution only by a homogeneous solution, as it must. Discuss 'causality', *i.e.* mysterious influences from the future.

#5. b, c. These show that translation becomes multiplication in Fourier space.

#7. Once (or even before!) you've solved, answer this: how is the solution $u(x, t)$ at time t related to the solution for the case $c = 0$ at the same time t ? [Hint: the previous question is useful here]

C) Use the *sifting property*

$$\int_{-\infty}^{\infty} \delta(x - a) f(x) dx = f(a)$$

to find the Fourier transform of the delta distribution $\delta(x - a)$. Now write the inversion formula—this gives you a new and useful representation of the delta distribution. By interchanging the labels x and ξ, deduce the Fourier transform of the plane wave function e^{ikx}. Add your answer to Table 6.2.

#10. [Hint: write out $|\hat{u}(\xi)|^2 = \hat{u}(\xi)\overline{\hat{u}(\xi)}$ using a double integral, use the above, then simplify]. This is the continuous analogue of Parseval's equality on p. 213. The Fourier transform is a (continuous rather than countably infinite) orthogonal expansion.

#11.

#15. I suggest you don't use the hint until you have a convolution expression for $u(x, y)$ as in Example 6.35, of which you may piggyback off the final result. The problem corresponds to injecting current density into the edge of a resisitive medium and solving for the voltage field—a useful medical imaging technique (Electrical Impedance Tomography).