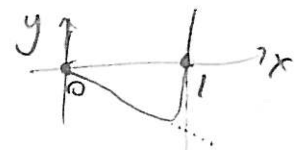
Barnett
6/1/07

SOLUTIONS

Math 46: Applied Math: Final

3 hours, 80 points total, 10 questions worth wildly varying numbers of points

1. [9 points] Use singular perturbation methods to find a uniform approximate solution to the boundary-value problem
$$\varepsilon y'' - 2y' - e^y = 0, \qquad \varepsilon \ll 1, \qquad y(0) = 0, \qquad y(1) = 0$$
As always, remember to check and explain the location of any boundary layer(s).



OUTER (set $\varepsilon = 0$): $-2y_0' - e^{y_0} = 0 \Rightarrow -\frac{1}{e^{y_0}}\frac{dy_0}{dx} = \frac{1}{2}$

$\Rightarrow -\int e^{-y_0}\,dy_0 = \int \frac{1}{2}\,dx$

$\Rightarrow e^{-y_0} = \frac{x}{2} + c \xrightarrow{\text{take log}} y_0(x) = -\ln\left(\frac{x}{2} + c\right)$

INNER: try BL @ $x=0$: $\xi = \frac{x}{\delta}$, $Y(\xi) = y(x)$ get $\frac{\varepsilon}{\delta^2}Y'' - \frac{2}{\delta}Y' - e^Y = 0$

Dominant balance: power of δ, power of ε; slope $= -1$ so $\delta^{-1}\varepsilon = 1$, $\delta = \varepsilon$. subst. $\delta = \varepsilon$, divide by ε.

$Y'' - 2Y' + \cancel{\varepsilon e^Y} = 0$ ignore at leading order.

But this has ~~growing~~ exponential only: $Y(\xi) = Ae^{+2\xi} + B$

$\Rightarrow$ not valid BL.

$\Rightarrow$ BL @ $x=1$, $\xi = \frac{1-x}{\delta}$ everything same as above except $\frac{\varepsilon}{\delta^2}Y'' + \frac{2}{\delta}Y' - e^Y = 0$

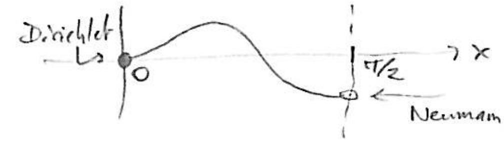
$\Rightarrow$ inner gen. soln. $Y_i(\xi) = \underbrace{Ae^{-2\xi} + B}_{\text{must have } A=-B \text{ to match BC at } x=1.}$, $\lim_{\xi\to\infty} Y(\xi) = B = c_m$

Common limit c_m must match outer $y_0(x) = -\ln\left(\frac{x}{2} + c\right)$ need $c=1$ to make outer match BC @ $x=0$.

$c_m = \lim_{x\to 1} y_0(x) = -\ln\left(\frac{1}{2}+1\right) = -\ln 3/2$

$$y_u(x) = y_e(x) + Y_i\left(\frac{x}{\varepsilon}\right) \underbrace{- c_m}_{\text{cancels } B \text{ in } Y_i} = -\ln\left(\frac{x}{2}+1\right) + \left(\ln\frac{3}{2}\right)e^{-2x/\varepsilon}$$

2. [9 points] Consider the differential operator $Ly := -y'' - 4y$ acting on functions obeying *mixed* boundary conditions $y(0) = 0$ and $y'(\pi/2) = 0$ (this might arise for an elastic string stretched over a frictionless hill, fixed at one end and free at the other).



(a) Find the complete set of eigenvalues and eigenfunctions of L.

$Ly = \lambda y$ so $-y'' - 4y = \lambda y$

$\Rightarrow y'' + (\lambda + 4)y = 0$

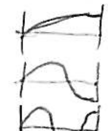
$\Rightarrow$ gen. soln. $y(x) = A \sin\sqrt{\lambda+4}\,x + B\cos\sqrt{\lambda+4}\,x$

$B \to 0$ to obey BC @ $x=0$.

@ $x = \pi/2$ must be $(2n+1)\frac{\pi}{2}$ to satisfy Neumann BC.

so $\sqrt{\lambda+4}\,\frac{\pi}{2} = (2n+1)\frac{\pi}{2}$ or $\lambda_n = (2n+1)^2 - 4$ for $n = 0, 1, 2 \ldots$

with $y_n(x) = \sin(2n+1)x$ eigenfunctions. (unnormalized)



(b) Find the Green's function for the inhomogeneous problem $Lu = f$.

Either use eigenfunction expansion $g(x,\xi) = \sum_{n=0}^{\infty} \frac{y_n(x)\,y_n(\xi)}{\lambda_n}$

since y_n's needed to be normalized.

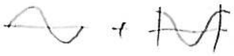
$= \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)x \; \sin(2n+1)\xi}{(2n+1)^2 - 4}$

OR solve $Ly = 0$ to get $-y'' - 4y = 0$ ie $y(x) = A\sin 2x + B\cos 2x$

$u_1(0) = 0$ so $u_1(x) = \sin 2x$

$u_2'(1) = 0$ so $u_2(x) = \cos 2x$

$W = 2\cos^2 2x - (-2\sin^2 2x) = 2.$



$p(\xi) = 1$.

so $g(x,\xi) = -\frac{1}{2}\begin{cases} \sin 2x \, \cos 2\xi & x < \xi \\ \sin 2\xi \, \cos 2x & x > \xi \end{cases}$

(c) What is the *lowest* derivative (zeroth, first, second,...) of the Green's kernel $g(x,\xi)$ that is discontinuous?

[$g(x,\xi)$ has a 'kink' in both x & ξ, at $x = \xi$.] obviously this is not clear from eigenfunction expansion, but is from direct solution, or past experience.

(d) [BONUS:] What is the *spectrum* of the Green's operator $Gu(x) := \int_0^{\pi/2} g(x,\xi)u(\xi)d\xi$?

$G = L^{-1}$ (that's the point of creating it, since $Lu = f$ solved by $u = L^{-1}f = Gf$)

so spectrum of G is the set $\{\lambda_n^{-1}\}_{n=0,1,\ldots}$ Check $Ly_n = \lambda_n y_n$

so $(GL)y_n = y_n = \lambda_n G y_n$

3. [6 points] Prove that eigenfunctions (with different eigenvalues) of the Laplace operator in a bounded domain Ω, with homogeneous Neumann boundary conditions ($\partial u/\partial n = 0$ on $\partial\Omega$) are *orthogonal* on the domain.

this says u, v are eigenfunctions of Δ with eigenvalues λ, μ respectively

$$\Delta u = \lambda u$$
$$\Delta v = \mu v$$

mult. v

mult. u

$$u \Delta v = \mu u v$$
$$v \Delta u = \lambda u v.$$

subtract & integrate

$$\underbrace{\int_\Omega (u \Delta v - v \Delta u)\, dx}_{\text{Green's 2nd Identity.}} = (\mu - \lambda) \underbrace{\int_\Omega u v\, dx}_{\text{is inner product } (u,v)}$$

$$\int_{\partial\Omega} \Big(u \underbrace{\frac{\partial v}{\partial n}}_{0} - v \underbrace{\frac{\partial u}{\partial n}}_{0} \Big) dA = 0$$

so $\quad (\mu - \lambda)(u, v) = 0$

if $\mu \neq \lambda$ then $(u, v) = 0$, $\underline{u \text{ orthogonal to } v \text{ on } \Omega.}$

QED.

[BONUS:] The above $\lambda = 0$ eigenfunction has a simple form. Use it to prove that a necessary condition for existence of a solution to the Neumann problem

$$\begin{aligned} \Delta u &= 0 && \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= f && \text{on } \partial\Omega \end{aligned}$$

is that the average value of f on the boundary is zero.

namely the constant function eg. $v \equiv 1$ with eigenvalue $\mu = 0$.

Green's 2nd Identity:

$$\int_\Omega \Big(u \underbrace{\cancel{\Delta v}}_{\substack{0 \text{ since} \\ v \text{ const.}}} - v \underbrace{\cancel{\Delta u}}_{\substack{0 \text{ since} \\ u \text{ harmonic in } \Omega.}} \Big) dx = \int_{\partial\Omega} \Big(u \underbrace{\cancel{\frac{\partial v}{\partial n}}}_{\substack{0 \text{ since} \\ v \text{ const.}}} - v \underbrace{\boxed{\frac{\partial u}{\partial n}}}_{= f\text{, the BVP.}} \Big) dA$$

so $\quad 0 = \int_{\partial\Omega} 1 \cdot f\, dA = \int_{\partial\Omega} f\, dA \qquad$ QED.

4. [8 points] Consider the integral operator $Ku(x) := \int_0^1 (x - 3y)u(y)dy$. [Hint: what type of integral operator is it?] $\longrightarrow$ degenerate Fredholm.

$$K(x,y) = \sum_{j=1}^{2} \alpha_j(x)\beta_j(y)$$

(a) Find the eigenvalues of K, and their multiplicities.

$\alpha_1(x) = x$ $\quad$ $\beta_1(y) = 1$

$\alpha_2(x) = 1$ $\quad$ $\beta_2(y) = -3y$

2×2 matrix

$$A_{ij} = (\beta_i, \alpha_j) = \begin{pmatrix} \int_0^1 x \cdot 1 dx & \int_0^1 1 \cdot 1\, dx \\ \int_0^1 -3x \cdot x\, dx & \int_0^1 -3x\, dx \end{pmatrix} = \begin{pmatrix} 1/2 & 1 \\ -1 & -3/2 \end{pmatrix}$$

Find λ's:

$$|A - \lambda I| = (\tfrac{1}{2} - \lambda)(-\tfrac{3}{2} - \lambda) + 1 = \lambda^2 + \lambda + \tfrac{1}{4} = (\lambda + \tfrac{1}{2})^2 = 0 \quad \text{so } \lambda = -\tfrac{1}{2} \text{ twice.}$$

Eigenvalues of K: $\lambda = -1/2$ (twice), $\lambda = 0$ (∞ multiplicity)

(b) Find an eigenfunction of K corresponding to a nonzero eigenvalue.

$$A - \lambda I = A + \tfrac{1}{2} I = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

has only a 1-dimensional nullspace spanned by $\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \vec{c}$ eigenvector.

Eigenfunction of K is $\sum_{j=1}^{2} \alpha_j(x) c_j = 1 - x$

(c) Is $Ku(x) + \frac{1}{2}u(x) = 1$ (the constant function) soluble? Why? (Don't solve)

$+1/2$ is same as $-\lambda$ for the $\lambda = -1/2$ eigenvalue.

So $Ku - \lambda u = 1$ only soluble if 1 in Range of $Ku - \lambda u$

$\hookrightarrow$ equiv. to $A\vec{c} - \lambda \vec{c} = \vec{F}$ $\longrightarrow$ the vector $(\beta_j, f) = \begin{pmatrix} 1 \\ -3/2 \end{pmatrix}$

But $\vec{F}$ is not in Col A = Span$\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\}$. $\Rightarrow$ no soln

(d) Is $Ku(x) + u(x) = 1$ soluble? Why? (Don't solve)

$\lambda = -1$.

Yes since $-1 \neq$ eigenvalue, the solution exists and is unique, for any RHS function f.

I gave first 4 for free given my typo, which no-one caught.

I'm sorry: I omitted the u_t term here: should be $-\Delta u + u_t = f$

5. [6 points] Use an energy argument to prove *uniqueness* for the solution to the inhomogeneous heat equation

$$\begin{aligned} -\Delta u(\mathbf{x},t) &= f(\mathbf{x},t) \qquad \mathbf{x}\in\Omega,\ t>0,\\ u(\mathbf{x},t) &= g(\mathbf{x}) \qquad \mathbf{x}\in\partial\Omega,\\ u(\mathbf{x},0) &\equiv 0 \qquad \mathbf{x}\in\Omega, \end{aligned}$$

makes inhomogeneous.

in a bounded domain $\Omega \subset \mathbb{R}^n$, where $f(\mathbf{x},t)$ is a heat source term and $g(\mathbf{x})$ is an imposed boundary temperature distribution.

Say u_1, u_2 are solutions, then $w := u_1 - u_2$ satisfies

$$\begin{cases} \Delta w = w_t & \leftarrow \text{note now it's homogeneous heat eqn.}\\ w = 0 & \text{on } \partial\Omega \\ w = 0 & \text{for } t=0, \text{ in } \Omega. \end{cases}$$

We need to show $w(\vec{x},t) \equiv 0$ is the unique solution.

$$E(t) := \int_\Omega w(\vec{x},t)^2 dx \qquad \text{so} \qquad E'(t) = 2\int_\Omega w w_t \, dx$$

$$= 2\int_\Omega w \Delta w \, dx = \underbrace{-2\int_\Omega \nabla w \cdot \nabla w \, dx}_{\geq 0} + 2\int_{\partial\Omega} \underbrace{w}_{0} \frac{\partial w}{\partial n} \, dA \leq 0$$

But $E(0) = \int 0^2 dx = 0$

and $E(t) \geq 0$ so $E(t) = 0 \ \forall t > 0$, so $w \equiv 0$, $u_1 = u_2$ unique.

6. [5 points] Find the convolution of the function $e^{-x^2/2a^2}$ with the function $e^{-x^2/2b^2}$ preferably by using Fourier transforms. (You have just shown how standard deviations add for statistically-independent normal variables!)

$$u_a(x) := e^{-\left(\frac{1}{2a^2}\right)x^2} \xrightarrow{\ F\ } \sqrt{2\pi}\, a\, e^{-\frac{a^2\xi^2}{2}} = \hat{u}_a(\xi)$$

the "a" from table of FTs.

product by convolution thm.

$$\text{so} \quad u_a * u_b \xrightarrow{\ F\ } \hat{u}_a(\xi)\,\hat{u}_b(\xi) = 2\pi ab\, e^{-(a^2+b^2)\frac{\xi^2}{2}}$$

now inverse transform back to get real-space func. Use "a" from table $= \dfrac{1}{2(a^2+b^2)}$

$$\sqrt{\frac{2\pi a^2b^2}{a^2+b^2}}\, e^{-\frac{x^2}{2(a^2+b^2)}} \xleftarrow{\ F^{-1}\ } \sqrt{2\pi\frac{a^2b^2}{a^2+b^2}} \cdot \sqrt{\pi 2(a^2+b^2)}\, e^{-\frac{\xi^2}{4}2(a^2+b^2)}$$

so standard deviations a & b gave $\sqrt{a^2+b^2}$ (Add in quadrature).

7. [13 points] Consider the perturbed initial-value problem for $y(t)$ on $t > 0$,

$$y'' + y = 4\varepsilon y(y')^2, \qquad \varepsilon \ll 1, \qquad y(0) = 1, \qquad y'(0) = 0$$

(a) Find a 2-term asymptotic approximation using regular perturbation theory. [Hints: You may find the power-reduction identities on the last page useful. You will get partial credit for leaving the 2^{nd} term as the solution to a clearly-specified IVP.]

resonant freqs' $\alpha = \pm i$. $(e^{\alpha t})$.

Unperturbed $\quad y_0'' + y_0 = 0 \qquad$ so $\quad y_0(t) = \underbrace{A}_{1}\cos t + \underbrace{B}_{y'(0)=0}\sin t$.

$$= \cos t$$

$$(y_0'' + \varepsilon y_1'' + \dots) + (y_0 + \varepsilon y_1 + \dots) = 4\varepsilon(y_0 + \dots)(y_0' + \dots)^2$$

$O(\varepsilon^1)$: $\quad y_1'' + y_1 = 4 y_0 y_0'^2 = 4\cos t(-\sin t)^2$

$$= \cos t - \cos 3t$$

power-reduction identities.

with ICs $\quad y_1(0) = 0, \quad y_1'(0) = 0$.

Solve this IVP: driving term has on-resonance part $\cos t$ (but $\cos 3t$ is off-resonance)

Method of Undetermined Coeffs try $Y(t) = At\sin t + B\cos 3t$

$Y' = A\sin t + At\cos t - 3B\sin 3t$

so

$Y'' + Y = 2A\cos t - 8B\cos 3t$. $\qquad Y'' = A\cos t + A\cos t - At\sin t - 9B\cos 3t$

match driving so $2A = 1$, $A = 1/2$

and $-8B = -1$, $B = 1/8$ $\Big\}$ so $Y(t) = \frac{1}{2}t\sin t + \frac{1}{8}\cos 3t$

Gen. Soln $\quad y_1(t) = Y(t) + c_1\cos t + c_2\sin t \qquad$ need $\left.\begin{matrix} c_1 = -1/8 \\ c_2 = 0 \end{matrix}\right\}$ to match ICs, both 0.

$$= \frac{1}{2}t\sin t + \frac{1}{8}(\cos 3t - \cos t)$$

So $\quad y_r(t) = \cos t + \varepsilon\left[\frac{t\sin t}{2} + \frac{\cos 3t - \cos t}{8}\right] + O(\varepsilon^2)\dots$

(b) Is this a uniform approximation for $t \in (0, \infty)$? Why?

No, since $\varepsilon t \sin t$ is unbounded for any fixed ε, as $t \to \infty$.

(c) Use the Poincaré-Lindstedt method to give a more useful 2-term approximation. [Hint: rescale to $\tau = \omega t$ where ω is perturbed from the value 1]

$$\omega = 1 + \varepsilon \omega_1 + \cdots$$

rescale time:

$$\omega^2 y'' + y = 4\varepsilon y (\omega y')^2$$

where prime means $\frac{d}{d\tau}$

$$(1 + \varepsilon\omega_1 \cdots)^2 (y_0'' + \varepsilon y_1'' + \cdots) + y_0 + \varepsilon y_1 + \cdots = 4\varepsilon (y_0 + \varepsilon y_1 \cdots)(1 + \varepsilon\omega_1 + \cdots)^2 (y_0' + \varepsilon y_1' + \cdots)^2$$

$O(\varepsilon^0)$ same as before, giving $y_0(\tau) = \cos\tau$

$O(\varepsilon^1)$ gives

$$\underbrace{2\omega_1 y_0''}_{2\omega_1(-\cos\tau)} + y_1'' + y_1 = 4 y_0 y_0'^2 = \cos\tau - \cos 3\tau$$

as before

so

$$y_1'' + y_1 = \underbrace{2\omega_1 \cos\tau + \cos\tau}_{\text{make cancel out if } \omega_1 = -1/2} - \cos 3\tau$$

$$= -\cos 3\tau$$

which has solution as before:

$$y_1(\tau) = \frac{1}{8}(\cos 3\tau - \cos\tau)$$

so

$$y_a(t) = \cos\tau + \frac{\varepsilon}{8}(\cos 3\tau - \cos\tau) + O(\varepsilon^2) \ldots$$

where

$$\tau = \left(1 - \varepsilon/2 + O(\varepsilon^2) \ldots\right) t .$$

(d) Is this a uniform approximation for $t \in (0, \infty)$?

yes... that's the point. (no secular terms)

This was supposed to be a challenging question, but combines techniques you know.

8. [8 points] Consider the 1D wave equation $u_{tt} = c^2 u_{xx}$ in $x \in \mathbb{R}$, $t > 0$.

(a) Use the method of Fourier transforms to write a general solution $u(x,t)$ [Hint: when it comes to writing an ODE solution, use complex exponentials]

$$u_{tt} = c^2 u_{xx} \xrightarrow{\mathcal{F} \text{ in } x} \hat{u}_{tt}(\xi,t) = c^2(-i\xi)^2 \hat{u}(\xi,t)$$

$\Rightarrow$ $\hat{u}_{tt} + (c\xi)^2 \hat{u} = 0$, solve the ODE in t (for fixed ξ).

$$\hat{u}(\xi,t) = \hat{a}(\xi)e^{ic\xi t} + \hat{b}(\xi)e^{-ic\xi t}$$

unknown functions.

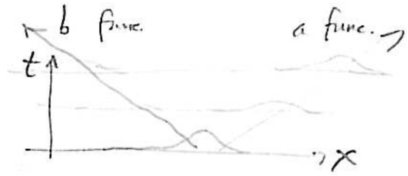
note I suggested you write this rather than $\hat{a}(\xi)\sin(c\xi t) + \hat{b}(\xi)\cos(c\xi t)$

Fourier transform back & use convolution thm.

$$u(x,t) = a * \mathcal{F}^{-1}(e^{ict\xi}) + b * \underbrace{\mathcal{F}^{-1}(e^{-ict\xi})}_{\text{look in table, is } \delta(x+ct)}$$

$$= \int a(x-y)\,\delta(y-ct)\,dy + \int b(x-y)\,\delta(y+ct)\,dy$$

$$= a(x-ct) + b(x+ct)$$



two traveling general functions.

(b) Use this to find the solution given 'displacement' initial conditions $u(x,0) = f(x)$ and $u_t(x,0) \equiv 0$.

Match in Fourier space:

$$\hat{f}(\xi) = \hat{a}(\xi) + \hat{b}(\xi)$$

$$0 = ic\xi\,\hat{a}(\xi) - ic\xi\,\hat{b}(\xi)$$

so $\hat{b}(\xi) = +\hat{a}(\xi)$

so $\hat{a}(\xi) = \hat{b}(\xi) = \dfrac{\hat{f}(\xi)}{2}$

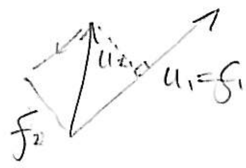
$$\Rightarrow \quad u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right]$$

$u_1 \quad u_2$

9. [5 points] Consider the set of two functions $\{1, x\}$ on the interval $x \in [0,1]$.

   (a) Replace the second function by one which turns this into an *orthogonal set.*

   in the span of $\{1, x\}$, I meant (This is reasonably clear by context).



Subtract $\frac{(u_1, u_2)}{\|u_1\|^2} u_1$ from u_2 to orthogonalize it against u_1.

$$f_2 = u_2 - \frac{\int_0^1 u_1 u_2 \, dx}{\int_0^1 u_1^2 \, dx} \cdot u_1 = x - \frac{\int_0^1 1 \cdot x \, dx}{\int_0^1 1^2 \, dx} 1$$

$$= x - 1/2$$

so $\{\underset{f_1}{1}, \underset{f_2}{x - 1/2}\}$ are orthog. set.

   (b) Find the best approximation (in the mean-square or $L^2[a,b]$ sense) to the function x^2 using this orthogonal set.

$f(x)$

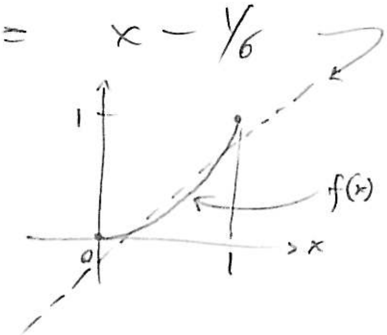
minimum mean-square error given by $\sum_{j=1}^{2} c_j f_j(x)$ eg see §4.1

with $c_j = \frac{(f_j, f)}{\|f_j\|^2}$

$$c_1 = \frac{\int_0^1 1 \cdot x^2 \, dx}{\int_0^1 1^2 \, dx} = 1/3$$

$$c_2 = \frac{\int_0^1 (x - 1/2) x^2 \, dx}{\int_0^1 (x - 1/2)^2 \, dx} = \frac{\frac{1}{4} - \frac{1}{2} \cdot \frac{1}{3}}{\int_{-1/2}^{1/2} y^2 \, dy} = \frac{1/12}{1/12} = 1$$

$$\sum_{j=1}^{2} c_j f_j(x) = 1/3 + (x - 1/2) = x - 1/6$$



10. [11 points] Short-answer questions—do give a brief explanation if asked for.

(a) The frequency f of a sinusoidal deep-water wave is related only to its wavelength λ and the acceleration due to gravity g. What does dimensional analysis tell you about this relation?

$$\begin{array}{cc} & \begin{array}{ccc} f & \lambda & g \end{array} \\ \begin{array}{c} L \\ T \end{array} & \left[\begin{array}{ccc} 0 & 1 & 1 \\ -1 & 0 & -2 \end{array}\right] \end{array} \qquad \Pi_1 = \frac{g}{\lambda f^2} = \text{const. by Buckingham Pi Thm.}$$

$$\text{or } f = c\sqrt{\frac{g}{\lambda}}.$$

(b) Compute the Fourier transform of the 'one-sided exponential' $u(x) = \begin{cases} e^{-ax} & x \geq 0 \\ 0 & x < 0 \end{cases}$

$$\hat{u}(\xi) := \int_{-\infty}^{\infty} e^{ix\xi} u(x)\,dx = \int_0^{\infty} e^{-ax} e^{ix\xi}\,dx$$

$$= \int_0^{\infty} e^{(i\xi - a)x}\,dx = \frac{1}{i\xi - a}[0-1] = \frac{1}{a - i\xi}$$

(c) Does a solution to $\int_0^1 \sin x \sin y\, u(y) dy = x^2$ exist? Is it unique? Why?

No. $\underbrace{\int_0^1 \sin x \sin y\, u(y) dy}$ is always of the form $c \sin x$ (c const.) so cannot equal x^2.

x^2 is not in Range of operator, which is Span$\{\sin x\}$

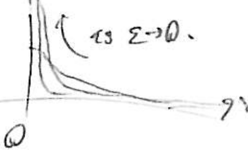
(d) Can a Green's function exist for the ODE problem $Ly := -y'' = f$ with *periodic* boundary conditions $y(0) = y(1)$ and $y'(0) = y'(1)$? Why?

Greens func. exists iff 0 is not an eigenvalue of L.

$Ly = 0$ ie $y'' = 0$ ie $y(x) = Ax + B$

BCs give $B = A + B$ so $A = 0$, & $A = A$. Any B gives soln. $\Rightarrow$ nontrivial $\Rightarrow \lambda = 0$ is eigenvalue $\Rightarrow$ no Greens func.

(e) Is $\frac{\varepsilon}{\varepsilon^2+x^2}$ pointwise convergent to zero in $x \in (0, \infty)$? Is it uniformly convergent in this same interval? Explain.



is pointwise convergent since for any fixed $x \in (0, \infty)$,

$$\frac{\varepsilon}{\varepsilon^2 + x^2} \to 0 \text{ as } \varepsilon \to 0.$$

is not uniformly convergent since first taking $x \to 0$, $\frac{\varepsilon}{\varepsilon^2+x^2} \to \frac{1}{\varepsilon}$ which is unbounded as $\varepsilon \to 0$.