# Math 46: Applied Math: Final

3 hours, 80 points total, 10 questions worth wildly varying numbers of points

(post-exam typo-corrected version)

1. [9 points] Use singular perturbation methods to find a uniform approximate solution to the boundary-value problem
$$\varepsilon y'' - 2y' - e^y = 0, \qquad \varepsilon \ll 1, \qquad y(0) = 0, \qquad y(1) = 0$$
As always, remember to check and explain the location of any boundary layer(s).

2. [9 points] Consider the differential operator $Ly := -y'' - 4y$ acting on functions obeying *mixed* boundary conditions $y(0) = 0$ and $y'(\pi/2) = 0$ (this might arise for an elastic string stretched over a frictionless hill, fixed at one end and free at the other).

   (a) Find the complete set of eigenvalues and eigenfunctions of $L$.

   (b) Find the Green's function for the inhomogeneous problem $Lu = f$.

   (c) What is the *lowest* derivative (zeroth, first, second,...) of the Green's kernel $g(x, \xi)$ that is discontinuous?

   (d) [BONUS:] What is the *spectrum* of the Green's operator $Gu(x) := \int_0^{\pi/2} g(x, \xi) u(\xi) d\xi$ ?

3. [6 points] Prove that eigenfunctions (with different eigenvalues) of the Laplace operator in a bounded domain $\Omega$, with homogeneous Neumann boundary conditions ($\partial u/\partial n = 0$ on $\partial\Omega$) are *orthogonal* on the domain.

[BONUS:] The above $\lambda = 0$ eigenfunction has a simple form. Use it to prove that a necessary condition for existence of a solution to the Neumann problem

$$\begin{aligned} \Delta u &= 0 \qquad \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= f \qquad \text{on } \partial\Omega \end{aligned}$$

is that the average value of $f$ on the boundary is zero.

4. [8 points] Consider the integral operator $Ku(x) := \int_0^1 (x - 3y)u(y)dy$. [Hint: what type of integral operator is it?]

   (a) Find the eigenvalues of $K$, and their multiplicities.

   (b) Find an eigenfunction of $K$ corresponding to a nonzero eigenvalue.

   (c) Is $Ku(x) + \frac{1}{2}u(x) = 1$ (the constant function) soluble? Why? (Don't solve)

   (d) Is $Ku(x) + u(x) = 1$ soluble? Why? (Don't solve)

5. [6 points] Use an energy argument to prove *uniqueness* for the solution to the inhomogeneous heat equation

$$\begin{aligned} -\Delta u(\mathbf{x},t) + u_t &= f(\mathbf{x},t) \qquad \mathbf{x}\in\Omega,\ \ t>0, \\ u(\mathbf{x},t) &= g(\mathbf{x}) \qquad\quad \mathbf{x}\in\partial\Omega, \\ u(\mathbf{x},0) &\equiv 0 \qquad\qquad \mathbf{x}\in\Omega, \end{aligned}$$

in a bounded domain $\Omega\subset\mathbb{R}^n$, where $f(\mathbf{x},t)$ is a heat source term and $g(\mathbf{x})$ is an imposed boundary temperature distribution.

6. [5 points] Find the convolution of the function $e^{-x^2/2a^2}$ with the function $e^{-x^2/2b^2}$ preferably by using Fourier transforms. (You have just shown how standard deviations add for statistically-independent normal variables!)

7. [13 points] Consider the perturbed initial-value problem for $y(t)$ on $t > 0$,

$$y'' + y = 4\varepsilon y(y')^2, \qquad \varepsilon \ll 1, \qquad y(0) = 1, \qquad y'(0) = 0$$

(a) Find a 2-term asymptotic approximation using regular perturbation theory. [Hints: You may find the power-reduction identities on the last page useful. You will get partial credit for leaving the $2^{nd}$ term as the solution to a clearly-specified IVP.]

(b) Is this a uniform approximation for $t \in (0, \infty)$ ? Why?

(c) Use the Poincaré-Lindstedt method to give a more useful 2-term approximation. [Hint: rescale to $\tau = \omega t$ where $\omega$ is perturbed from the value 1]

(d) Is this a uniform approximation for $t \in (0, \infty)$ ?

8. [8 points] Consider the 1D wave equation $u_{tt} = c^2 u_{xx}$ in $x \in \mathbb{R}$, $t > 0$.

(a) Use the method of Fourier transforms to write a general solution $u(x,t)$ [Hint: when it comes to writing an ODE solution, use complex exponentials]

(b) Use this to find the solution given 'displacement' initial conditions $u(x,0) = f(x)$ and $u_t(x,0) \equiv 0$.

9. [5 points] Consider the set of two functions $\{1, x\}$ on the interval $x \in [0, 1]$.

   (a) Replace the second function by one in $\text{Span}\{1, x\}$ which turns this into an *orthogonal set.*

   (b) Find the best approximation (in the mean-square or $L^2[a, b]$ sense) to the function $x^2$ using this orthogonal set.

10. [11 points] Short-answer questions—do give a brief explanation if asked for.

(a) The frequency $f$ of a sinusoidal deep-water wave is related only to its wavelength $\lambda$ and the acceleration due to gravity $g$. What does dimensional analysis tell you about this relation?

(b) Compute the Fourier transform of the 'one-sided exponential' $u(x) = \begin{cases} e^{-ax} & x \geq 0 \\ 0 & x < 0 \end{cases}$

(c) Does a solution to $\int_0^1 \sin x \, \sin y \, u(y) dy = x^2$ exist? Is it unique? Why?

(d) Can a Green's function exist for the ODE problem $Ly := -y'' = f$ with *periodic* boundary conditions $y(0) = y(1)$ and $y'(0) = y'(1)$? Why?

(e) Is $\frac{\varepsilon}{\varepsilon^2+x^2}$ pointwise convergent to zero in $x \in (0, \infty)$ ? Is it uniformly convergent in this same interval? Explain.

## Useful formulae

Stationary phase ($c$ = interior maximum of $g$)

$$\int f(x)e^{\lambda g(x)}dx \approx f(c)e^{\lambda g(c)}\sqrt{\frac{-2\pi}{\lambda g''(c)}}$$

Binomial

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

Error function [note $\mathrm{erf}(0) = 0$ and $\lim_{z\to\infty}\mathrm{erf}(z) = 1$]:

$$\mathrm{erf}(z) := \frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}ds$$

Euler relations

$$e^{i\theta} = \cos\theta + i\sin\theta, \qquad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

Power-reduction identities

$$\begin{aligned}
\cos^3\theta &= \frac{1}{4}(3\cos\theta + \cos 3\theta) \\
\cos^2\theta\sin\theta &= \frac{1}{4}(\sin\theta + \sin 3\theta) \\
\cos\theta\sin^2\theta &= \frac{1}{4}(\cos\theta - \cos 3\theta) \\
\sin^3\theta &= \frac{1}{4}(3\sin\theta - \sin 3\theta)
\end{aligned}$$

Fourier Transforms:

| $u(x)$ | $\hat{u}(\xi)$ |
|---|---|
| $\delta(x-a)$ | $e^{ia\xi}$ |
| $e^{ikx}$ | $2\pi\delta(k+\xi)$ |
| $e^{-ax^2}$ | $\sqrt{\frac{\pi}{a}}e^{-\xi^2/4a}$ |
| $e^{-a\|x\|}$ | $\frac{2a}{a^2+\xi^2}$ |
| $H(a-\|x\|)$ | $2\frac{\sin(a\xi)}{\xi}$ |
| $u^{(n)}(x)$ | $(-i\xi)^n\hat{u}(\xi)$ |
| $u * v$ | $\hat{u}(\xi)\hat{v}(\xi)$ |