MATH 46 WORKSHEET : Volterra Integral Eqns

Barnett
5/4/07

A) Solve for $u(t)$ in the equation for stock decay with kernel $k(t)=e^{-bt}$:

$$a\,k(t) \quad + \quad \int_0^t k(t-\tau)\,u(\tau)\,d\tau \quad = a$$

B) Differentiate to solve the following integral eqn (assume f' exists, $a\neq 0$):

$$\int_0^t y\,u(y)\,dy \quad - \quad a\,u(t) \quad = \quad f(t) \qquad \text{on } 0\le t\le 1.$$

[convert to an ODE]

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~ SOLUTIONS ~

A) Solve for $u(t)$ in the equation for stock decay with kernel $k(t) = e^{-bt}$:

$$a\,k(t) + \int_0^t k(t-\tau)\,u(\tau)\,d\tau = a$$

$$a e^{-bt} + \underbrace{\int_0^t e^{-b(t-\tau)}u(\tau)\,d\tau}_{e^{-bt}\int_0^t e^{b\tau}u(\tau)\,d\tau} = a$$

Mult. by e^{bt}:

$$a + \int_0^t e^{b\tau}u(\tau)\,d\tau = a e^{bt}$$

$\frac{d}{dt}$

$$e^{bt}u(t) = b a e^{bt} \quad \Rightarrow \quad \boxed{u(t) = ab}$$

B) Differentiate to solve the following integral eqn (assume f' exists, $a \neq 0$):

$$\int_0^t y\,u(y)\,dy - a\,u(t) = f(t) \qquad \text{on } 0 \le t \le 1.$$

[convert to an ODE] $\frac{d}{dt}$

$$t\,u(t) - a\,u'(t) = f'(t)$$

so

$$u'(t) \underbrace{- \frac{t}{a}}_{p(t)} u(t) = \underbrace{-\frac{f'(t)}{a}}_{q(t)}$$

in linear 1st order ODE with IC: $u(0) = \frac{-f(0)}{a}$

Integrating factor is $e^{\int p\,dt} = e^{-\frac{1}{2a}t^2}$

so

$$\left(e^{-t^2/2a}u\right)' = -\frac{1}{a}f'$$

$$e^{-t^2/2a}u = -\frac{1}{a}\int f'\,dt = -\frac{f}{a} + c$$

$c = 0$ to match IC.

$$u(t) = -\frac{f(t)}{a}e^{t^2/2a}$$

This is #11 on p. 245