MATH 46 WORKSHEET : Review phase plane

4/4/07
Barnett

A) Show that the ODE $\begin{cases} x' = y \\ y' = -x \end{cases}$ is equivalent to $x'' + x = 0$

therefore write down general solution:

Is the critical point stable?

locally asymptotically stable?

B) Reverse the above to turn $mx'' = f(x, x')$

Newton's 2^{nd} Law with force dep. on position & vel.

into a 1^{st} ~~order~~ coupled system:

If $E := \frac{1}{2}my^2 + V(x)$

Take $\frac{d}{dt}E$ and show it vanishes when $f(x,x') = f(x) = -\frac{dV}{dx}$

C) $\binom{1}{1}$ is a critical point of $\begin{cases} x' = y - x \\ y' = -y + \frac{5x^2}{4+x^2} \end{cases}$

Linearize about this critical point, ie get a 2×2 (Jacobean) matrix A

[Hint: $x = 1 + \alpha$, etc...]

What type of motion happens here?

What is stability?

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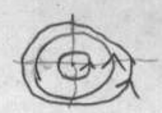
A) Show that the ODE $\begin{cases} x' = y \\ y' = -x \end{cases}$ is equivalent to $x'' + x = 0$

subst. $y' = x''$ into 2nd eqn.

therefore write down general solution: $x(t) = c_1 \sin t + c_2 \cos t$



Is the critical point stable? yes

locally asymptotically stable? no.

B) Reverse the above to turn $mx'' = f(x, x')$ Newton's 2nd Law with force dep. on position & vel.

into a ~~1st order~~ coupled system: ← see p. 58.

$x' = y$ as before

$y' = x'' = \frac{1}{m} f(x, x')$ ← but have to convert to y!

Ie $\begin{cases} x' = y \\ y' = \frac{1}{m} f(x,y) \end{cases}$ (*)

If $E := \frac{1}{2} m y^2 + V(x)$

Take $\frac{d}{dt}E$ and show it vanishes when $f(x,y) = f(x) = -\frac{dV}{dx}$

$$\frac{d}{dt}\left(\frac{1}{2} m y^2 + V(x)\right) = \frac{m}{2} 2 y y' + \overset{y}{x'} \frac{dV}{dx}$$

$$= y\left(m y' + \frac{dV}{dx}\right) = m y \Big(\underbrace{y' - \tfrac{1}{m} f(x,y)}_{\text{zero by } (*)}\Big) = 0$$

C) $\binom{1}{1}$ is a critical point of $\begin{cases} x' = y - x \\ y' = -y + \frac{5x^2}{4+x^2} \end{cases}$

Linearize about this critical point, ie get a 2×2 (Jacobean) matrix A

[Hint: $x = 1+\alpha$, etc..] sub $\begin{cases} x = 1+\alpha \\ y = 1+\beta \end{cases}$ so $\alpha' = (1+\beta) - (1+\alpha)$

$\beta' = -(1+\beta) + \frac{5(1+\alpha)^2}{4+(1+\alpha)^2}$

Ie $\alpha' = -\alpha + \beta$

$\beta' = -\beta \quad -1 + \boxed{\frac{5 + 10\alpha + O(\alpha^2)}{4 + 1 + 2\alpha + O(\alpha^2)}}$ So $A = \begin{pmatrix} -1 & 1 \\ 8/5 & -1 \end{pmatrix}$

$\frac{1 + 2\alpha + \cdots}{1 + \frac{2}{5}\alpha + \cdots} = 1 + \frac{8}{5}\alpha \ldots$

What type of motion happens here? }
What is stability? } discuss in class.