Math 46: Applied Math: Final

3 hours, 80 points total, 10 questions worth wildly varying numbers of points

(post-exam typo-corrected version)

1. [9 points] Use singular perturbation methods to find a uniform approximate solution to the boundary-value problem

$$\varepsilon y'' - 2y' - e^y = 0, \qquad \varepsilon \ll 1, \qquad y(0) = 0, \qquad y(1) = 0$$

As always, remember to check and explain the location of any boundary layer(s).

2. [9 points] Consider the differential operator $Ly := -y'' - 4y$ acting on functions obeying *mixed* boundary conditions $y(0) = 0$ and $y'(\pi/2) = 0$ (this might arise for an elastic string stretched over a frictionless hill, fixed at one end and free at the other).

   (a) Find the complete set of eigenvalues and eigenfunctions of L.

   (b) Find the Green's function for the inhomogeneous problem $Lu = f$.

   (c) What is the *lowest* derivative (zeroth, first, second,...) of the Green's kernel $g(x, \xi)$ that is discontinuous?

   (d) [BONUS:] What is the *spectrum* of the Green's operator $Gu(x) := \int_0^{\pi/2} g(x, \xi)u(\xi)d\xi$?

3. [6 points] Prove that eigenfunctions (with different eigenvalues) of the Laplace operator in a bounded domain Ω, with homogeneous Neumann boundary conditions ($\partial u/\partial n = 0$ on $\partial\Omega$) are *orthogonal* on the domain.

[BONUS:] The above $\lambda = 0$ eigenfunction has a simple form. Use it to prove that a necessary condition for existence of a solution to the Neumann problem

$$\begin{aligned} \Delta u &= 0 \qquad \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= f \qquad \text{on } \partial\Omega \end{aligned}$$

is that the average value of f on the boundary is zero.

4. [8 points] Consider the integral operator $Ku(x) := \int_0^1 (x - 3y)u(y)dy$. [Hint: what type of integral operator is it?]

(a) Find the eigenvalues of K, and their multiplicities.

(b) Find an eigenfunction of K corresponding to a nonzero eigenvalue.

(c) Is $Ku(x) + \frac{1}{2}u(x) = 1$ (the constant function) soluble? Why? (Don't solve)

(d) Is $Ku(x) + u(x) = 1$ soluble? Why? (Don't solve)

5. [6 points] Use an energy argument to prove *uniqueness* for the solution to the inhomogeneous heat equation

$$\begin{aligned} -\Delta u(\mathbf{x},t) + u_t &= f(\mathbf{x},t) \qquad \mathbf{x} \in \Omega, \quad t > 0, \\ u(\mathbf{x},t) &= g(\mathbf{x}) \qquad \mathbf{x} \in \partial\Omega, \\ u(\mathbf{x},0) &\equiv 0 \qquad \mathbf{x} \in \Omega, \end{aligned}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, where $f(\mathbf{x},t)$ is a heat source term and $g(\mathbf{x})$ is an imposed boundary temperature distribution.

6. [5 points] Find the convolution of the function $e^{-x^2/2a^2}$ with the function $e^{-x^2/2b^2}$ preferably by using Fourier transforms. (You have just shown how standard deviations add for statistically-independent normal variables!)

7. [13 points] Consider the perturbed initial-value problem for $y(t)$ on $t > 0$,

$$y'' + y = 4\varepsilon y(y')^2, \qquad \varepsilon \ll 1, \qquad y(0) = 1, \qquad y'(0) = 0$$

(a) Find a 2-term asymptotic approximation using regular perturbation theory. [Hints: You may find the power-reduction identities on the last page useful. You will get partial credit for leaving the 2^{nd} term as the solution to a clearly-specified IVP.]

(b) Is this a uniform approximation for $t \in (0, \infty)$? Why?

(c) Use the Poincaré-Lindstedt method to give a more useful 2-term approximation. [Hint: rescale to $\tau = \omega t$ where ω is perturbed from the value 1]

(d) Is this a uniform approximation for $t \in (0, \infty)$?

8. [8 points] Consider the 1D wave equation $u_{tt} = c^2 u_{xx}$ in $x \in \mathbb{R}$, $t > 0$.

(a) Use the method of Fourier transforms to write a general solution $u(x, t)$ [Hint: when it comes to writing an ODE solution, use complex exponentials]

(b) Use this to find the solution given 'displacement' initial conditions $u(x, 0) = f(x)$ and $u_t(x, 0) \equiv 0$.

9. [5 points] Consider the set of two functions $\{1, x\}$ on the interval $x \in [0, 1]$.

(a) Replace the second function by one in $\text{Span}\{1, x\}$ which turns this into an *orthogonal set.*

(b) Find the best approximation (in the mean-square or $L^2[a, b]$ sense) to the function x^2 using this orthogonal set.

10. [11 points] Short-answer questions—do give a brief explanation if asked for.

(a) The frequency f of a sinusoidal deep-water wave is related only to its wavelength λ and the acceleration due to gravity g. What does dimensional analysis tell you about this relation?

(b) Compute the Fourier transform of the 'one-sided exponential' $u(x) = \begin{cases} e^{-ax} & x \geq 0 \\ 0 & x < 0 \end{cases}$

(c) Does a solution to $\int_0^1 \sin x \, \sin y \, u(y) dy = x^2$ exist? Is it unique? Why?

(d) Can a Green's function exist for the ODE problem $Ly := -y'' = f$ with *periodic* boundary conditions $y(0) = y(1)$ and $y'(0) = y'(1)$? Why?

(e) Is $\frac{\varepsilon}{\varepsilon^2 + x^2}$ pointwise convergent to zero in $x \in (0, \infty)$? Is it uniformly convergent in this same interval? Explain.

Useful formulae

Stationary phase (c = interior maximum of g)

$$\int f(x)e^{\lambda g(x)}dx \approx f(c)e^{\lambda g(c)}\sqrt{\frac{-2\pi}{\lambda g''(c)}}$$

Binomial

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

Error function [note $\text{erf}(0) = 0$ and $\lim_{z\to\infty}\text{erf}(z) = 1$]:

$$\text{erf}(z) := \frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}ds$$

Euler relations

$$e^{i\theta} = \cos\theta + i\sin\theta, \qquad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

Power-reduction identities

$$\begin{aligned}
\cos^3\theta &= \frac{1}{4}(3\cos\theta + \cos 3\theta) \\
\cos^2\theta\sin\theta &= \frac{1}{4}(\sin\theta + \sin 3\theta) \\
\cos\theta\sin^2\theta &= \frac{1}{4}(\cos\theta - \cos 3\theta) \\
\sin^3\theta &= \frac{1}{4}(3\sin\theta - \sin 3\theta)
\end{aligned}$$

Fourier Transforms:

$u(x)$	$\hat{u}(\xi)$
$\delta(x-a)$	$e^{ia\xi}$
e^{ikx}	$2\pi\delta(k+\xi)$
e^{-ax^2}	$\sqrt{\frac{\pi}{a}}e^{-\xi^2/4a}$
$e^{-a\|x\|}$	$\frac{2a}{a^2+\xi^2}$
$H(a-\|x\|)$	$2\frac{\sin(a\xi)}{\xi}$
$u^{(n)}(x)$	$(-i\xi)^n\hat{u}(\xi)$
$u*v$	$\hat{u}(\xi)\hat{v}(\xi)$