Consider the error in using an N-term sum with coeffs $c_n = (f_n, f)$

$$E_N = f - \sum_{n=1}^{N} c_n f_n$$

including the products of sums, and use formula for c_n, then simplify.

Write out $\|E_N\|^2$, expand <u>as much as possible</u>, and finally use $\|E_N\|^2 \geq 0$:

BONUS : Use your above simplest expression for $\|E_N\|^2$ to show, for any coeffs a_n:

$$\|f - \sum_{n=1}^{N} a_n f_n\|^2 - \|E_N\|^2 = \sum_{n=1}^{N} (a_n - c_n)^2 \geq 0$$

so for any $\{a_n\}$ the error cannot do better than $\{c_n\}$, these c_n are <u>optimal</u>.

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$$\|E_N\|^2 = (E_N, E_N) = \left(f - \sum c_n f_n ,\; f - \sum c_n f_n\right)$$

by linearity $(a, b+c) = (a,b) + (a,c)$

$$= (f,f) - 2\left(f, \sum c_n f_n\right) + \left(\sum c_n f_n ,\; \sum c_n f_n\right)$$

$$= \|f\|^2 - 2\sum c_n \underbrace{(f, f_n)}_{= c_n} + \sum_n \sum_m c_n c_m \underbrace{(f_n, f_m)}_{\begin{cases} 1 & \text{if } m=n \\ 0 & \text{otherwise}\end{cases}}$$

$$= \|f\|^2 - 2\sum c_n^2 + \sum c_n^2$$

$$= \|f\|^2 - \sum_{n=1}^{N} c_n^2$$

≥ 0 since $\|\cdot\|^2$ anything is non-negative! so $\sum_{n=1}^{N} c_n^2 \leq \|f\|^2$ Bessel's Inequality QED.

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See book p. 212