Math 46: Applied Math: Homework 9

due Wed May 30—despite appearances, a bit shorter to let you study for final

I think you'll find this an enlightening voyage of discovery.

p.365-367: #13. (review) Nice that you can prove uniqueness for certain *nonlinear* PDEs too.

A): This question connects the theory you've learned to applied acoustics. To a good approximation (see Ch. 7 if interested), sound waves in air obey the wave equation

$$u_{tt} = c^2 \Delta u$$

where $u(\mathbf{x}, t)$ is the pressure in space and time, with homogeneous Neumann boundary conditions $\partial u/\partial n = 0$ on solid walls. The speed of sound is $c = 340$ m/s.

1. Show that time-harmonic waves of the form $u(\mathbf{x}, t) = U(\mathbf{x}) \sin(2\pi f t)$ satisfy the eigenfunction relation $-\Delta U = \lambda U$, known as the Helmholtz equation. What is λ in terms of the frequency f? (If you're interested, the wavenumber is $\sqrt{\lambda}$, equal to 2π divided by wavelength).
2. Find a formula for eigenfunctions $U_{m,n}(\mathbf{x})$ and eigenvalues $\lambda_{m,n}$, for $m, n = 0, 1, \ldots$, in the closed domain Ω being a 2D rectangular box of side lengths M and N. [Hint: similar to Example 6.21. Does the 1D situation have a zero eigenvalue?]
3. Guess (or work out) the generalization to 3D for a cuboid with sides L, N and M. Write a formula for the corresponding frequency $f_{l,m,n}$. Use this 3D formula to compute the smallest five nonzero resonant frequencies of a shower cubicle of dimensions $2 \times 1 \times 1$ m^3. You have just computed musical pitches which resonate when you sing in the shower!
4. *Estimate* the number of such resonant frequencies lying in the range of human hearing, *i.e.* frequencies from (essentially) zero up to 2×10^4 Hz. [Hint: Use intuition & please discuss together. Is there a geometric interpretation of your formula for $f_{l,m,n}$?]

p.395-398: #4. As a function of ξ this is called a Cauchy distribution. It comes up in statistics and has an infinite variance.

#5. b, c.

#7. Once (or even before!) you've solved, answer this: how is the solution $u(x, t)$ at time t related to the solution for the case $c = 0$ at the same time t ?

B): Use the *sifting property*

$$\int_{-\infty}^{\infty} \delta(x - a) f(x) dx = f(a)$$

to find the Fourier transform of the delta distribution $\delta(x - a)$. Now write the inversion formula—this gives you a new and useful representation of the delta distribution. By interchanging the labels x and ξ, deduce the Fourier transform of the plane wave function e^{ikx}. Add your answer to Table 6.2.

#10. [Hint: write out $|\hat{u}(\xi)|^2 = \hat{u}(\xi)\overline{\hat{u}(\xi)}$ using a double integral, use the above, then simplify]. This is the continuous analogue of Parseval's equality on p. 213. The Fourier transform is a (continuous rather than countably infinite) orthogonal expansion.

#11.

#15. I suggest you don't use the hint until you have a convolution expression for $u(x, y)$ as in Example 6.35, of which you may piggyback off the final result. The problem corresponds to injecting current density into the edge of a resisitive medium and solving for the voltage field—a useful medical imaging technique (Electrical Impedance Tomography).