Zeros

Definition

If f is analytic at z_0, then we say that f has a zero of order $m \geq 1$ at z_0 if

$$0 = f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0)$$

and $f^{(m)}(z_0) \neq 0$. If $f^{(n)}(z_0) = 0$ for all $n \geq 0$, then we call z_0 a zero of infinite order.

Theorem

If f is analytic in a domain D and if f has a zero of infinite order in D, then f is identically zero.

Isolated Zeros

Theorem

Suppose that f is a non-constant analytic function on a domain D. If $z_0 \in D$ is a zero of f, then z_0 has finite order $m \geq 1$ and there is an analytic function g on D such that $g(z_0) \neq 0$ and

$$f(z) = (z - z_0)^m g(z) \quad \text{for all } z \in D.$$

Corollary

Suppose that f is a non-constant analytic function on a domain D. Then the zeros of f are isolated. That is, if $z_0 \in D$ and $f(z_0) = 0$, then there is a $r > 0$ such that

$$f(z) \neq 0 \quad \text{if } z \in B'_r(z_0).$$

An Example

Example

Note that

$$g(z) = \sin\left(\frac{\pi}{z}\right)$$

is analytic in $D = B_1(1)$. But $g\left(\frac{1}{n}\right) = 0$ for all $n \geq 1$. Since $0 \notin D$, all of the zeros of g are isolated.

Isolated Singularities

Definition

Suppose that f is analytic in $B'_R(z_0)$ for some $R > 0$. The we call z_0 an isolated singularity for f.

Remark (Key Remark)

Suppose that f has an isolated singularity at z_0. Then f has a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j}$$

converging in some $B'_R(z_0)$ with $R > 0$. The coefficients a_n and b_j depend only on f and are given by the formulas in Laurent's Theorem.

Flavors of Isolated Singularities

Definition

Suppose that f has an isolated singularity at z_0 with associated Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j} \quad \text{for } z \in B'_R(z_0)$$

for $R > 0$.

1. If $b_j = 0$ for all j, then we call z_0 a removable singularity.
2. If $b_m \neq 0$ and $b_j = 0$ for all $j > m$, then we call z_0 a pole of order m.
3. If there are infinitely many j such that $b_j \neq 0$, then we call z_0 an essential singularity.

Remark

Note that an isolated singularity must be exactly one of these three types.

Classifying Removable Singularities

Theorem

Suppose that f has an isolated singularity at z_0. Then the following are equivalent.

1. *z_0 is a removable singularity for f.*
2. *We can define, or re-define if necessary, $f(z_0)$ so that f is analytic at z_0.*
3. *$\lim_{z \to z_0} f(z)$ exists (∞ NOT allowed).*
4. *f is bounded near z_0; that is, there is a $M > 0$ and a $r > 0$ such that $|f(z)| \leq M$ if $z \in B_r'(z_0)$.*