Maximum Modulus

Theorem (Maximum Modulus Principle)

Suppose that f is analytic on a domain D. If there is point $z_0 \in D$ such that

$$|f(z)| \leq |f(z_0)| \quad \text{for all } z \in D,$$

then f is constant.

Remark (Bounded Regions)

- Recall that a domain D is bounded if there is a $R > 0$ such that $D \subset B_R(0)$.
- The boundary ∂D of D is the set of points z such that every open ball $B_r(z)$ contains points in D and not in D.
- The closure $\overline{D}$ of D is the union of D and ∂D. Of course, $\overline{D}$ is closed.
- If D is bounded, then $\overline{D}$ is closed and bounded.
- Thus if D is a bounded domain, then any continuous real-valued function on $\overline{D}$ must attain its maximum and minimum on $\overline{D}$.

Bounded Domains

Theorem

Suppose that D is a bounded domain with closure $\overline{D} = D \cup \partial D$. Suppose also that $f : \overline{D} \subset \mathbf{C} \to \mathbf{C}$ is continuous and analytic on D. Then $|f(z)|$ attains is maximum on ∂D.

Remark

In the text, the authors describe the hypotheses above by saying that "f is analytic on a bounded domain D and continuous up to and including its boundary". In class, we will use the formalism above.

Series of Complex Numbers

Definition

A series of complex numbers is an expression of the form

$$\sum_{n=0}^{\infty} c_n \tag{1}$$

with each $c_n \in \mathbf{C}$. The n^{th} partial sum of (1) is

$$s_n = c_0 + c_1 + \cdots + c_n.$$

We say that the series converges to $s \in \mathbf{C}$ if $\lim_{n\to\infty} s_n$ exists and equals s. Otherwise we say that the series diverges.

Reducing to Calculus II

Lemma

We have

$$\sum_{n=0}^{\infty} c_n = s$$

if and only if

$$\sum_{n=0}^{\infty} \mathrm{Re}(c_n) = \mathrm{Re}(s) \quad \textit{and} \quad \sum_{n=0}^{\infty} \mathrm{Im}(c_n) = \mathrm{Im}(s).$$

Standard Tests for Convergence

Theorem (Comparison Test)

Suppose that

$$|c_n| \leq a_n \quad \textit{for all } n \geq N.$$

Then if

$$\sum_{n=0}^{\infty} a_n < \infty,$$

the complex series

$$\sum_{n=0}^{\infty} c_n$$

converges.

Absolute Convergence

Definition

The series

$$\sum_{n=0}^{\infty} c_n$$

converges absolutely if

$$\sum_{n=0}^{\infty} |c_n| < \infty.$$

Remark

Notice that by the comparison test, an absolutely convergent series is convergent.

The Ratio Test

Theorem (Ratio Test)

Suppose that

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L.$$

If $L < 1$, *then*

$$\sum_{n=0}^{\infty} c_n$$

converges absolutely. If $L > 1$, *then the series diverges.*

Remark

Notice that having the limit exist is part of the hypotheses. In general, the limit might not exist. Also, if $L = 1$, the test gives no information.

Our Favorite Series

Theorem (Geometric Series)

The geometric series

$$\sum_{n=0}^{\infty} ac^n = a + ac + ac^2 + \cdots$$

converges to

$$\frac{a}{1-c}$$

if $|c| < 1$. *Otherwise the series diverges.*