The Index

Definition

If Γ is a (not necessarily simple) closed contour and $a \notin \Gamma$, then the index of Γ about a is

$$\mathrm{Ind}_\Gamma(a) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{z - a}\, dz.$$

Remark

We proved that $\mathrm{Ind}_\Gamma(a)$ is always an integer. Drawing pictures and using the Deformation Invariance Theorem helps to convince us that $\mathrm{Ind}_\Gamma(a)$ counts the number of times Γ wraps around a in a counterclockwise direction.

In a Different Course

I our course, we happily assume we can invoke the Jordan Curve Theorem at will. We did this for our version of the Cauchy Integral Formula. A more modest approach would use the index.

Theorem (The Cauchy Integral Formula)

Suppose that f is analytic in a simply connected domain D and that Γ is a closed contour in D. Then for any $z \in D \setminus \Gamma$,

$$\mathrm{Ind}_\Gamma(z) f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\omega)}{\omega - z}\, d\omega.$$

Remark

Before anyone asks, you're not responsible for this.

More Good Homework

Theorem (EP-2)

Suppose that f is analytic on and inside a simple closed contour Γ and that f has no zeros on Γ. We (can) assume that f has only finitely many zeros inside Γ. Let N_f be the number of zeros of f inside of Γ counted up to multiplicity. Then

$$N_f = \frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)}\, dz.$$

Even Better

Theorem

Suppose that f is analytic on and inside a simple closed contour Γ and that f has no zeros on Γ. Let $f(\Gamma)$ be the contour $\{ f(z) : z \in \Gamma \}$. Then

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)}\, dz = N_f = \mathrm{Ind}_{f(\Gamma)}(0).$$

Walking the Dog

Theorem (Walking the Dog Lemma)

Let Γ_0 and Γ_1 be closed contours parameterized by $z_k : [0,1] \to \mathbf{C}$ with $k = 0$ and $k = 1$, respectively. Suppose that for some $a \in \mathbf{C}$ we have

$$\left|z_0(t) - z_1(t)\right| < \left|z_0(t) - a\right| \quad \textit{for all } t \in [0,1].$$

Then

$$\operatorname{Ind}_{\Gamma_0}(a) = \operatorname{Ind}_{\Gamma_1}(a).$$

Remark

This says that if I walk Willy around the Green so that Willy is always closer to me than I am to the bonfire, then Willy and I circle the bonfire the same number of times.

Rouché's Theorem

Theorem (Rouché's Theorem)

Suppose that f and g are analytic on and inside a simple closed contour Γ and that

$$|f(z) - g(z)| < |f(z)| \quad \text{for all } z \in \Gamma.$$

Then, up to multiplicity, f and g have the same number of zeros inside Γ.

The Final Exam

1. The final is Friday from 3pm to 6pm in 006 Kemeny.
2. The exam covers everything we have covered in lecture and in homework with the exception of indented contours as in Section 6.5 of the text.
3. This includes today's lecture which is based on Section 6.7 in the text.
4. As in just about any mathematics course, the final exam will emphasize material towards the end of the course:
    1. Zinn's Law
    2. Exponential Growth
    3. Interest