Riemann's Theorem

Theorem (Riemann's Theorem)

Suppose that g is continuous on a contour Γ. Let $D = \{\, z \in \mathbf{C} : z \notin \Gamma \,\}$. For each $n = 1, 2, 3, \dots,$ define

$$F_n(z) = \int_\Gamma \frac{g(\omega)}{(\omega - z)^n}\, d\omega \quad \text{for } z \in D.$$

Then F_n is analytic on D and for each n,

$$F_n'(z) = nF_{n+1}(z) = n \int_\Gamma \frac{g(\omega)}{(\omega - z)^{n+1}}\, d\omega.$$

Remark

Several people asked for a reference for the proof. I included a link to my proof on the web page.

Moving On

Theorem (Morera's Theorem)

Suppose that f is continuous on a domain D and that for all closed contours Γ in D we have

$$\int_\Gamma f(z)\,dz = 0.$$

The f is analytic on D.

Theorem (Cauchy's Estimates)

Suppose that f is analytic on $B_R(z_0)$ and that $|f(z)| \leq M$ for all $z \in B_R(z_0)$. Then for $n = 0, 1, 2, \ldots,$ we have

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!M}{R^n}.$$

Liouville's Theorem

Theorem (Liouville's Theorem)

A bounded entire function must be constant.

Proof.

Suppose that f is entire with $|f(z)| \leq M$ for all $z \in \mathbf{C}$. Fix $z_0 \in \mathbf{C}$. If $R > 0$, the f is analytic on $B_R(z_0)$ so Cauchy's Estimates imply that

$$|f'(z_0)| \leq \frac{M}{R}.$$

But we can take R as large as we like. Hence $f'(z_0) = 0$. Since z_0 is arbitrary, this implies $f' \equiv 0$. Thus f must be constant. □

More Good Stuff

Theorem (Extreme Value Theorem)

A continuous real-valued function on a closed and bounded subset E of $\mathbf{R}^2$ attains its maximum and minimum on E.

Theorem (Fundamental Theorem of Algebra)

Suppose that $p(z)$ is a polynomial with complex coefficients. If $\deg p(z) \geq 1$, then $p(z)$ has at least one complex root.