The Complex Exponential Function

Definition

For $z = x + iy$ we define

$$\exp(z) = e^z = e^x\big(\cos(y) + i\sin(y)\big)$$

Remark

For $\theta \in \mathbf{R}$, we have $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. So the basic properties of sin and cos imply that

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

This will be useful in problem #20 in §1.4.

Basic Properties

Lemma

For $z, w \in \mathbf{C}$,

1. $e^z e^w = e^{z+w}$, $e^{-z} = \frac{1}{e^z}$, *and* $\frac{e^z}{e^w} = e^{z-w}$.
2. *For all* $n \in \mathbb{Z}$, $(e^z)^n = e^{nz}$.

Corollary (De Moivre's Formula)

$$\big(\cos(\theta) + i\sin(\theta)\big)^n = \cos(n\theta) + i\sin(n\theta).$$

Nth Roots

Theorem

Every nonzero complex number $z = re^{i\theta}$ has exactly n distinct n^{th}-roots given by

$$w_k = \sqrt[n]{r}\exp\left(i\frac{\theta + 2\pi k}{n}\right) \quad \textit{for } k = 0, 1, \ldots, n-1.$$