Laurent Series I

Theorem (Laurent's Theorem)

Suppose that f is analytic in

$$A = \{ z \in \mathbf{C} : 0 \leq r < |z - z_0| < R \leq \infty \}$$

with $r < R$. Then there are complex constants a_n and b_j such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j}$$

for all $z \in A$. Moreover if C is any positively oriented simple closed contour in A with z_0 in its interior, then

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\omega)}{(\omega - z_0)^{n+1}} \, d\omega \quad \text{and}$$

$$b_j = \frac{1}{2\pi i} \int_C f(\omega)(\omega - z_0)^{j-1} \, d\omega.$$

Laurent Series II

Theorem

Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j} \tag{1}$$

converges in the annulus

$$A = \{ z \in \mathbf{C} : 0 \leq r < |z - z_0| < R \leq \infty \}$$

with $r < R$. Then f is analytic in A and (1) is the Laurent series for f in A. In particular, the coefficients a_n and b_j are given by the formulas on the previous slide.

Zeros

Definition

If f is analytic at z_0, then we say that f has a zero of order $m \geq 1$ at z_0 if

$$0 = f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0)$$

and $f^{(m)}(z_0) \neq 0$. If $f^{(m)}(z_0) = 0$ for all $m \geq 0$, then we call z_0 a zero of infinite order.

Theorem

Suppose that f is analytic in a domain D. If f has a zero of infinite order a some $z_0 \in D$, then f is identically zero in D.