Taylor's Theorem

Theorem (Taylor's Theorem)

Suppose that f is analytic in a disk $B_R(z_0)$ with $R > 0$. Then the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

for f about z_0 converges to $f(z)$ for all $z \in B_R(z_0)$. Furthermore the convergence is uniform in any subdisk

$$\overline{B_r(z_0)} = \{ z \in \mathbf{C} : |z - z_0| \leq r \}$$

provided $0 < r < R$.

Term-by-Term Differentiation

Theorem

Suppose that f is analytic in $B_R(z_0)$ with Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Then the Taylor series for the derivative f' in $B_R(z_0)$ is given by term-by-term differentiation:

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}.$$

Cauchy Product

Definition

The Cauchy Product of two Taylor series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n(z-z_0)^n$$

about z_0 is given by

$$\sum_{n=0}^{\infty} c_n(z-z_0)^n$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

$\tan(z)$

Example (MacLaurin Series for $\tan z$)

Here is

$$\tan(z) = z + \frac{z^3}{3} + \frac{2z^5}{15} + \frac{17z^7}{315} + \frac{62z^9}{2835} + \frac{1382z^{11}}{155925} + \frac{21844z^{13}}{6081075} + \frac{929569z^{15}}{638512875} + \dots$$