Maximum Modulus Theorem II

Theorem

Suppose that D is a bounded domain and that $f : \overline{D} \subset \mathbf{C} \to \mathbf{C}$ is continuous and analytic on D. Then $|f(z)|$ attains its maximum on the boundary ∂D of D.

Definition

Let $\{ F_n \}$ be a sequence of complex-valued functions on a set D.

1. We say that $\{ F_n \}$ converges pointwise to a function $F : D \subset \mathbf{C} \to \mathbf{C}$ if for all $z \in D$ we have
$$\lim_{n \to \infty} F_n(z) = F(z).$$
Thus for all $z \in D$ and $\epsilon > 0$ there is a $N = N(\epsilon, z)$ such that $n \geq N$ implies
$$\left| F(z) - F_n(z) \right| < \epsilon.$$

2. We say that $\{ F_n \}$ converges uniformly to a function $F : D \subset \mathbf{C} \to \mathbf{C}$ if for all $\epsilon > 0$ there is a $N = N(\epsilon)$ such that $n \geq N$ implies that
$$\left| F(z) - F_n(z) \right| < \epsilon \quad \text{for all } z \in D.$$

Definition

Let D be a set. We say a series

$$\sum_{n=0}^{\infty} f_n(z) \tag{1}$$

of functions $f : D \subset \mathbf{C} \to \mathbf{C}$ converges pointwise to a function F on a set D if the sequence of partial sums $\{ F_n \}$ given by

$$F_n(z) = \sum_{k=1}^{n} f_k(z)$$

converge pointwise F on D. Similarly, we say (1) converges uniformly to F on D if the partial sums $\{ F_n \}$ converge uniformly to F on D.

The Basic Example

Consider the series $\sum_{n=0}^{\infty} \left(\frac{z}{z_0}\right)^n$ with partial sums

$$F_n(z) = \sum_{k=0}^{n} \left(\frac{z}{z_0}\right)^k = \frac{\left(\frac{z}{z_0}\right)^{n+1} - 1}{\frac{z}{z_0} - 1}.$$

If we saw last time that F_n converges pointwise to

$$F(z) = \frac{1}{1 - \frac{z}{z_0}}.$$

on the disk $D = B_{|z_0|}(0)$.

Example Continued

We also saw that

$$\left|F(z) - F_n(z)\right| = \left| \frac{1}{1 - \frac{z}{z_0}} - \frac{\left(\frac{z}{z_0}\right)^{n+1} - 1}{\frac{z}{z_0} - 1} \right|$$

$$= \left| \frac{\left(\frac{z}{z_0}\right)^{n+1}}{1 - \frac{z}{z_0}} \right|$$

$$= \frac{\left|\frac{z}{z_0}\right|^{n+1}}{\left|1 - \frac{z}{z_0}\right|}$$