Residues

Definition

If f has an isolated singularity at z_0 with Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j}$$

for $z \in B'_R(z_0)$ with $R > 0$, then we call b_1 the residue of f at z_0 and write

$$\mathrm{Res}(f; z_0) = b_1.$$

Basics

Lemma

If f has a simple pole at z_0, then

$$\mathrm{Res}(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

Lemma (Basic Simple Pole Lemma)

If g and h are analytic at z_0 such that $g(z_0) \neq 0$ and such that h has a simple zero at z_0, then

$$f(z) := \frac{g(z)}{h(z)}$$

has a simple pole at z_0 and

$$\mathrm{Res}(f; z_0) = \frac{g(z_0)}{h'(z_0)}.$$

General Poles

Lemma

Suppose that f has a pole of order m at z_0. Then

$$\operatorname{Res}(f; z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left((z - z_0)^m f(z) \right).$$

Cauchy Residue Theorem

Theorem (Cauchy Residue Theorem)

Suppose that f is analytic on and inside a simple closed contour Γ except for isolated singularities at $z_1, \ldots, z_n$ inside of Γ. Then

$$\int_\Gamma f(z)\,dz = 2\pi i \sum_{k=1}^{m} \operatorname{Res}(f; z_k). \tag{1}$$

Remark (Notation)

We often write (1) as

$$\int_\Gamma f(z)\,dz = 2\pi i \sum_{z \text{ inside } \Gamma} \operatorname{Res}(f; z)$$

with the understanding that the sum is finite since $\operatorname{Res}(f; z) = 0$ if z is not a pole or essential singularity.

Trigonometric Integrals

- Observe that if we parameterize the positively oriented circle $|z| = 1$ by $z(t) = e^{i\theta}$ for $\theta \in [0, 1]$ then

$$\int_{|z|=1} F(z)\, dz = \int_0^{2\pi} F(e^{i\theta}) i e^{i\theta}\, d\theta.$$

- Furthermore, if $z = e^{i\theta}$ lies on the circle $|z| = 1$, then

$$\cos(\theta) = \frac{1}{2}\left(z + \frac{1}{z}\right) \quad \text{while} \quad \sin(\theta) = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$