Zeros

Theorem

Suppose that f is a non-constant analytic function on a domain D. If $z_0 \in D$ is a zero of f, then z_0 has finite order $m \geq 1$ and there is an analytic function g on D such that $g(z_0) \neq 0$ and

$$f(z) = (z - z_0)^m g(z) \quad \textit{for all } z \in D.$$

Corollary

Suppose that f is a non-constant analytic function on a domain D. Then the zeros of f are isolated. That is, if $z_0 \in D$ and $f(z_0) = 0$, then there is a $r > 0$ such that

$$f(z) \neq 0 \quad \textit{if } z \in B_r'(z_0).$$

Isolated Singularities

Definition

Suppose that f is analytic in $B'_R(z_0)$ for some $R > 0$. The we call z_0 an isolated singularity for f.

Remark (Key Remark)

Suppose that f has an isolated singularity at z_0. Then f has a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j}$$

converging in some $B'_R(z_0)$ with $R > 0$. The coefficients a_n and b_j depend only on f and are given by the formulas in Laurent's Theorem.

Flavors of Isolated Singularities

Definition

Suppose that f has an isolated singularity at z_0 with associated Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j} \quad \text{for } z \in B'_R(z_0)$$

for $R > 0$.

1. If $b_j = 0$ for all j, then we call z_0 a removable singularity.
2. If $b_m \neq 0$ and $b_j = 0$ for all $j > m$, then we call z_0 a pole of order m.
3. If there are infinitely many j such that $b_j \neq 0$, then we call z_0 and essential singularity.

Classifying Removable Singularities

Theorem

Suppose that f has an isolated singularity at z_0. Then the following are equivalent.

1. z_0 *is a removable singularity for* f.
2. *We can define, or re-define if necessary,* $f(z_0)$ *so that* f *is analytic at* z_0.
3. $\lim_{z \to z_0} f(z)$ *exists.*
4. f *is bounded near* z_0*; that is, there is a* $M > 0$ *and a* $r > 0$ *such that* $|f(z)| \leq M$ *if* $z \in B_r'(z_0)$.