Power Series 101

Theorem

Let

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

be a power series about z_0. *Then there is a* R *such that* $0 \leq R \leq \infty$ *and such that*

1. *The series converges absolutely if* $|z - z_0| < R$.
2. *The series converges uniformly on any subdisk*

   $$D_r = \{\, z \in \mathbf{C} : |z - z_0| \leq r \,\}$$

   provided that $0 < r < R$.
3. *The series diverges if* $|z - z_0| > R$.

Uniformly Good

Theorem

Suppose that $\{ f_n \}$ is a sequence of continuous complex-valued functions converging uniformly to f on a set D. Then f is continuous on D.

Theorem

Suppose that $\{ f_n \}$ is a sequence of continuous complex-valued functions converging uniformly to f on a set D. Then if Γ is any contour in D,

$$\int_\Gamma f(z)\,dz = \lim_{n\to\infty} \int_\Gamma f_n(z)\,dz.$$

Theorem

Suppose that $\{ f_n \}$ is a sequence of analytic complex-valued functions converging uniformly to f on a domain D. Then f is analytic on D.

Power Series vs. Taylor Series

Theorem

Suppose that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ has a positive radius of convergence $R > 0$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \tag{1}$$

is analytic in $D = B_R(z_0)$. Moreover

$$a_n = \frac{f^{(n)}(z_0)}{n!},$$

and (1) is the Taylor series for f about z_0.