Improper Integrals with Trigonometric Functions

Theorem (Improper Integrals Plus 1)

Suppose that $p(z)$ and $q(z)$ are polynomials with real coefficients such that

$$\deg p(z) + 1 \leq \deg q(z).$$

Let

$$F(z) = \frac{p(z)}{q(z)} e^{iaz}.$$

Then if $q(x)$ has no zeros on the real axis and $a > 0$ then

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos(ax)\, dx = \operatorname{Re}\Bigl(2\pi i \sum_{\operatorname{Im} z > 0} \operatorname{Res}(F; z)\Bigr)$$

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \sin(ax)\, dx = \operatorname{Im}\Bigl(2\pi i \sum_{\operatorname{Im} z > 0} \operatorname{Res}(F; z)\Bigr)$$

The Index

While we've been concentrating on definite and improper integrals, you have been doing some nice mathematics on homework!

Definition (The Index)

If Γ is a closed contour (not necessarily simple) and $a \notin \Gamma$, then the index of Γ about a is

$$\mathrm{Ind}_\Gamma(a) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{z-a}\, dz.$$

Remark

You proved on homework that $\mathrm{Ind}_\Gamma(a)$ is always an integer. Drawing pictures and using the Deformation Invariance Theorem helps to convince us that $\mathrm{Ind}_\Gamma(a)$ counts the number of times Γ wraps around a is a counterclockwise direction.

In a Different Course

Theorem (The Cauchy Integral Formula w/o the Jordan Curve Theorem)

Suppose that f is analytic in a simply connected domain D and that Γ is a closed contour in D. Then for any $z \in D \setminus \Gamma$,

$$\mathrm{Ind}_\Gamma(z) f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\omega)}{\omega - z}\, d\omega.$$

Remark

Before anyone asks, you're not responsible for this.

More Good Homework

Theorem (Homework)

Suppose that f is analytic on and inside a simple closed contour Γ and that f has no zeros on Γ. We (can) assume that f has only finitely many zeros inside Γ. Let N_f be the number of zeros of f inside of Γ counted up to multiplicity. Then

$$N_f = \frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)}\, dz.$$

Even Better

Theorem (Homework)

Suppose that f is analytic on and inside a simple closed contour Γ and that f has no zeros on Γ. Let $f(\Gamma)$ be the contour $\{ f(z) : z \in \Gamma \}$. Then

$$\frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)} \, dz = N_f = \operatorname{Ind}_{f(\Gamma)}(0).$$

Proof that $N_f = \text{Ind}_{f(\Gamma)}(0)$

Proof.

Suppose that Γ is parameterized by $z : [0,1] \to \mathbf{C}$ so that $f(\Gamma)$ is parameterized by $t \mapsto f(z(t))$ for $t \in [0,1]$. Then

$$\begin{aligned}
\text{Ind}_{f(\Gamma)}(0) &= \frac{1}{2\pi i} \int_{f(\Gamma)} \frac{1}{z}\, dz \\
&= \frac{1}{2\pi i} \int_0^1 \frac{1}{f(z(t))} f'(z(t)) z'(t)\, dt \\
&= \frac{1}{2\pi i} \int_0^1 \frac{f'(z(t))}{f(z(t))} z'(t)\, dt \\
&= \frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)}\, dz \\
&= N_f.
\end{aligned}$$

□

Walking the Dog

Theorem (Walking the Dog Lemma)

Let Γ_0 and Γ_1 be closed contours parameterized by $z_k : [0,1] \to \mathbf{C}$ with $k = 0$ and $k = 1$, respectively. Suppose that for some $a \in \mathbf{C}$ we have

$$\left|z_0(t) - z_1(t)\right| < \left|z_0(t) - a\right| \quad \text{for all } t \in [0,1].$$

Then

$$\mathrm{Ind}_{\Gamma_0}(a) = \mathrm{Ind}_{\Gamma_1}(a).$$

Remark

This says that if I walk Willy around the Green so that Willy is always closer to me than I am to the bonfire, then Willy and I circle the bonfire the same number of times.