Poles

Theorem

Suppose that f is analytic on a domain D except for an isolated singularity at z_0. The f has a pole of order $m \geq 1$ at z_0 if and only if there is an analytic function g on D with $g(z_0) \neq 0$ such that

$$f(z) = \frac{g(z)}{(z - z_0)^m} \quad \text{for all } z \in D \setminus \{ z_0 \}.$$

Theorem

Suppose that f has an isolated singularity at z_0. Then f has a pole at z_0 if and only if

$$\lim_{z \to z_0} |f(z)| = \infty.$$

Essential Singularities

Corollary

Suppose the f has an isolated singularity at z_0. Then f has an essential singularity at z_0 if and only if f is not bounded near z_0 and $\lim_{z\to z_0} |f(z)| \neq \infty$.

Theorem (Casorati-Weierstrass)

Suppose that f has an essential singularity at z_0. If $\epsilon > 0$, then

$$f\big(B'_\epsilon(z_0)\big) = \{\, f(z) : 0 < |z - z_0| < \epsilon \,\}$$

is dense in $\mathbf{C}$ *for all* $\epsilon > 0$.

Remark

We stated the Great and Little Picard Theorems for perspective only. They are not "legal tender" for homework and exams.

Last Time

Remark

We saw at the end of last lecture that if f has an isolated singularity at z_0 with Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j} \quad \text{for } z \in B'_R(z_0) \text{ with } R > 0,$$

then if Γ is any positively oriented simple closed contour in $B'_R(z_0)$ with z_0 in its interior we have

$$\int_\Gamma f(z)\, dz = 2\pi i b_1.$$

Note that this is just the formula for b_1 coming from Laurent's Theorem.