The Cauchy-Riemann Equations

Theorem (CR-I)

Suppose that $f(x+iy) = u(x,y) + iv(x,y)$ *is complex differentiable at* $z_0 = x_0 + iy_0$. *Then*

$$f'(z_0) = f_x(z_0) = -if_y(z_0).$$

In particular, both u *and* v *have partial derivatives at* (x_0, y_0) *and*

$$u_x(x_0,y_0) = v_y(x_0,y_0) \quad \text{and} \quad u_y(x_0,y_0) = -v_x(x_0,y_0). \tag{1}$$

Definition

We call (1) the Cauchy-Riemann Equations for f at $z_0 = x_0 + iy_0$.

Sufficient Conditions

Theorem (CR-II)

Suppose that $f(x+iy) = u(x,y) + iv(x,y)$ *is defined on* $D = B_r(z_0)$ *for some* $r > 0$. *Let* $z_0 = x_0 + iy_0$. *Suppose that*

1. u *and* v *have partial derivatives in* D.
2. *The partials of* u *and* v *are continuous at* (x_0, y_0).
3. *The Cauchy-Riemann equations hold for* f *at* (x_0, y_0).

Then $f'(z_0)$ *exists.*

Corollary

Suppose that $f(z) = u(z) + iv(z)$ *is defined on a domain* D *such that* u *and* v *have continuous partials throughout* D *and such that the Cauchy-Riemann equations for* f *hold throughout* D. *Then* f *is analytic on* D.