A Little Behind

I've fallen behind the posted homework assignments. So what was originally today's assignment is postponed until Monday, so this week's assignment—including this coming Monday's—will be due Wednesday. We'll meet in our x-hour next week (and the next), so the assignments for next week's Tuesday, Wednesday, and Friday will be the assignment for that week.

Cauchy Again

Theorem (The Cauchy Integral Formula)

Suppose that f is analytic on a simply connected domain D and that Γ is a simple closed contour in D. If z lies inside of Γ, then

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\omega)}{\omega - z}\, d\omega.$$

Remark

Things to keep in mind when trying to apply the Cauchy Integral Formula.

1. D must be simply connected.
2. Γ must be a simple closed contour.
3. f must be analytic on D.
4. z must be inside of Γ.

Reading the Textbook

Theorem (The Jordan Curve Theorem [NC-17])

If Γ is a simple closed contour in the plane, then the complement of Γ consists of two domains each of which has Γ as its boundary. One domain is unbounded. The bounded domain is called the interior of Γ. The interior is simply connected.

Remark

If Γ is a simple closed contour, we will take it on faith that if f is "analytic on and inside a simple closed contour Γ", then f is analytic on a simply connected domain D containing Γ. This allows the following corollaries of the Cauchy Integral Theorem (CIT) and the Cauchy Integral Formula (CIF).

Analytic on and Inside

Theorem

Suppose that f is analytic on and inside a simple closed contour Γ.

1. *[CIT] We have*
$$\int_\Gamma f(z)\,dz = 0.$$
2. *[CIF] If z lies inside Γ, then*
$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\omega)}{\omega - z}\,d\omega.$$

Smooth Functions

Writing higher derivatives gets difficult: f', f'', f''', $\ldots$. So in general, for $n \geq 1$ we write

$$\frac{d^n f}{dz^n}(x) := f^{(n)}(z).$$

And by convention,

$$f^{(0)}(z) = f(z).$$

Definition

Let D be a domain in $\mathbf{C}$. A function $f : D \subset \mathbf{C} \to \mathbf{C}$ is called C^∞ or smooth if the n^{th}-derivative $f^{(n)}(z)$ of f exists for all $n \geq 0$ and all $z \in D$. Similarly, a real-valued function $u : D \subset \mathbf{C} \to \mathbf{R}$ is called smooth if u has continuous partials of all orders in D.

Better than the Text

Theorem (Riemann's Theorem)

Suppose that g is continuous on a contour Γ. Let $D = \{ z \in \mathbf{C} : z \notin \Gamma \}$. For each $n = 1, 2, 3, \dots$, define

$$F_n(z) = \int_\Gamma \frac{g(\omega)}{(\omega - z)^n}\, d\omega \quad \text{for } z \in D.$$

Then F_n is analytic on D and for each n,

$$F_n'(z) = nF_{n+1}(z) = n \int_\Gamma \frac{g(\omega)}{(\omega - z)^{n+1}}\, d\omega.$$

The proof is tricky. We'll start by proving the result for $n = 1$. Then we'll show that if $n \geq 2$ and we know the result for $n - 1$, then it also holds for n. This will show that it holds for all n.

The Proof

1. Fix $z_0 \in D$.
2. Show that F_1 is continuous on D.
3. Define G_n and note that G_1 is continuous on D.
4. Show that $F_1'(z_0) = G_1(z_0) = F_2(z_0)$ (and hence $G_1'(z) = G_2(z)$).
5. Assume that for some $n \geq 2$ we know that $F_{n-1}'(z) = nF_n(z)$ (and hence $G_{n-1}'(z) = nG_n(z)$).
6. Observe that

$$F_n(z) - F(z_0) = G_{n-1}(z) - G_{n-1}(z_0) + (z - z_0)G_n(z_0)$$

7. Show F_n and G_n are continuous.
8. Finish by showing $F_n'(z_0) = nF_{n+1}(z_0)$.