Simply Connected

Definition

A domain D is called simply connected is every closed contour Γ in D can be continuously deformed to a point in D.

Examples

The whole complex plane $\mathbb{C}$ and any open disk $B_r(z_0)$ are simply connected. We'll see shortly that the annulus $A = \{ z \in \mathbb{C} : 1 < |z| < 2 \}$ is not simply connected.

The Deformation Invariance Theorem

Here is our first "major result". Note that I mistakenly called this "The Domain Invariance Theorem" on Monday.

Theorem (The Deformation Invariance Theorem)

Suppose that f is analytic is a domain D and that Γ_0 and Γ_1 are closed contours in D such that Γ_0 can be continuously deformed into Γ_1 inside of D. Then

$$\int_{\Gamma_0} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz.$$

The Cauchy Integral Theorem

Theorem (The Cauchy Integral Theorem)

Suppose that f is analytic in a simply connected domain D. Then for any closed contour Γ in D,

$$\int_\Gamma f(z)\,dz = 0.$$

Theorem

If f is analytic in a simply connected domain D, then f has an antiderivative in D.

Corollary

The annulus $A = \{\, z \in \mathbb{C} : 1 < |z| < 2 \,\}$ is not simply connected.

Corollary

If D is simply connected and $0 \notin D$, then there is an analytic branch of $\log z$ in D.