m42w06	Homework: 1	Due Date: 11/1/06

Metric spaces are a useful concept often used to bridge the gap between geometry and topology.

A metric space consists of a set X and a function $d\colon X \times X \to \mathbb{R}^1$ such that [1]

- $d(x, y) \geq 0$ with $d(x, y) = 0$ if and only if $x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) + d(y, z) \geq d(x, z)$.

The function d is refered to as a distance function or metric on X.

Problem 1A: [20 pts]

(**a**) Show that the $\mathbb{R}^3$ together with the function

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

is a metric space. (Try to argue geometrically rather than computationally).

A subset $X \subset \mathbb{R}^3$ is C^1 path-connected if for every pair of points $x, y \in X$[1], there is a C^1 curve $\gamma\colon [a, b] \to \mathbb{R}^3$ such that γ lies entirely in X and $\gamma(a) = x$, $\gamma(b) = y$. In other words, every pair of points can be connected by a C^1 curve.

For such a subset X, we can define a new function

$$d_X(x, y) = \min\left\{\int_a^b \|\dot{\gamma}(t)\| dt : \gamma \text{ is a } C^1 \text{ curve into } X \text{ with } \gamma(a) = x,\ \gamma(b) = y.\right\}$$

(**b**) Show that (X, d_X) is a metric space.

When $X = \{x_1^2 + x_2^2 + x_3^2 = 1\}$, i.e. the unit sphere centered at $(0, 0, 0)$, we shall see later that the length minimizing curves are the great circles (intersections of the sphere with planes passing through the origin)

(**c**) Use the fact that d_X is a metric space to prove the following result:

"The sum of any two angles formed at the vertex of a triangular pyramid is greater than the third angle."

(Hint: think about lengths of arcs of circles in terms of angles.)

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[1]The Cartesan product of sets $X \times Y$ consists of all pairs (x, y) such that $x \in X$ and $y \in Y$.

[1]The notation $x \in X$ mean x is an element of the set X.

Def: Given a point p contained in a subset X of $\mathbb{R}^2$ (or $\mathbb{R}^3$), the *tangent space to X at p*, denoted T_pX, is the collection of all pairs $\{(p, v)\}$ such that there exists a C^1 curve $\gamma\colon (-\epsilon, \epsilon) \to \mathbb{R}^2$ (or $\mathbb{R}^3$) with its image entirely in X such that

- $\gamma(0) = p$
- $\gamma'(0) = v$.

In other words (p, v) is in T_pX if there is a C^1 curve in X passing through p with tangent vector v at p. Note: with this definition, the curve must lie inside X defined on both sides of p.

The tangent space is defined in this slightly strange way to emphasize that the tangent vectors start at p, whereas the vectors v alone just represent a magnitude and direction and do not encode a position in space.

To help visualization, we often think about the *embedded tangent space*, denoted $\mathcal{E}_pX$ which consists of all points $\{p + v\}$ where $(p, v) \in T_pX$, i.e. the endpoints of all vectors v starting at p.

Problem 1B: [20 pts]

(**a**) Show that if $(p, v) \in T_pX$ and $\lambda \in \mathbb{R}$ then $(p, \lambda v) \in T_pX$.

(Note this means that the embedded tangent space is the union of $\{p\}$ with lines through p, although there maybe zero or infinitely many.)

Describe the embedded tangent spaces of the following subsets X at the given points p. Briefly justify your answers.

(**b**) $X = \{x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$, $p = (1, 0)$.

(**c**) $X = \{x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$, $p = (0, 0)$.

(**d**) $X = \{\max |x|, |y| = 1\} \subset \mathbb{R}^2$, $p = (1, 1)$.

(**e**) X is the image of the "figure-eight" curve $\gamma(t) = (\sin(2t), \sin t)$, $p = (0, 0)$.

(**f**) $X = \{z = x^2 + y^2\} \subset \mathbb{R}^3$, $p = (1, 0, 1)$.