I. Position, Velocity, Acceleration

(1) $a(t)=6t+4$ $\quad v(0)=-6\text{ cm/s}$ $\quad s(0)=9\text{ cm}$

Find $s(t)$.

$$v(t)=\int a(t)\,dt \qquad v(0)=-6 \Rightarrow C=-6$$

$$=\int 6t+4\,dt = 3t^2+4t+C$$

$$v(t)=3t^2+4t-6$$

$$s(t)=\int v(t)\,dt=\int 3t^2+4t-6\,dt = t^3+2t^2-6t+D$$

$$s(0)=9 \Rightarrow D=9$$

$$\boxed{s(t)=t^3+2t^2-6t+9}$$

(2) $a(t)=-9.8\text{ m/s}^2$ (we would give this to you on an exam)

$$v(t)=\int -9.8\,dt = -9.8t+C \qquad v(0)=48\text{ ft/s} \Rightarrow C=48$$

$$v(t)=-9.8t+48$$

$$s(t)=\int -9.8t+48\,dt = -4.9t^2+48t+D \qquad s(0)=432\text{ ft} \Rightarrow D=432$$

$$s(t)=-4.9t^2+48t+432$$

When does the ball reach its max height?

find when $v(t)=0$

$$-9.8t+48=0$$

$$t=\frac{48}{9.8}=\boxed{4.898\text{ s}} \text{ (Whoa!)}$$

When does the ball hit the ground?

Solve $s(t)=0$.

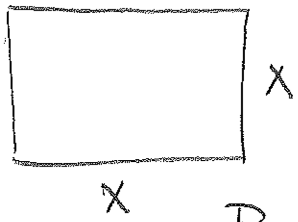
$$-4.9t^2+48t+432=0$$

quadratic formula: $t=\dfrac{-48\pm\sqrt{48^2-4\cdot(-4.9)(432)}}{(-4.9)(2)}$

$$=\underbrace{-56923}_{\text{this doesn't make sense.}} \text{ OR } \boxed{15.488\text{ s}}$$

2. Related Rates

(1)



$\frac{dx}{dt} = 0.2$ cm/s

P = perimeter $\quad \frac{dP}{dt}$ want

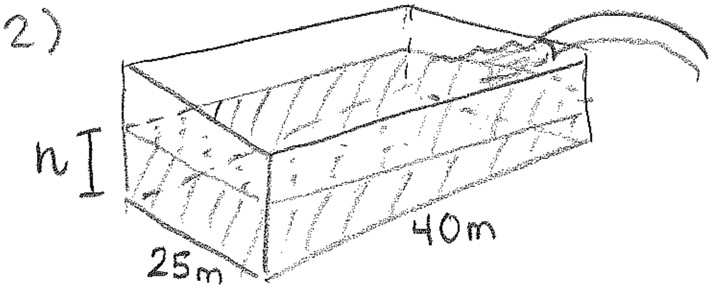
$P = 4x$ - relate variables

$\frac{d}{dt}(P) = \frac{d}{dt}(4x)$ - differentiate both sides

$$\frac{dP}{dt} = 4\frac{dx}{dt}$$

$$\frac{dP}{dt} = 4(0.2) = \boxed{0.8 \text{ cm/s}}$$

(2)



V = volume of water in pool

h = depth of water

$\frac{dV}{dt} = 500$ m³/min

$$V = 25 \cdot 40 \cdot h$$

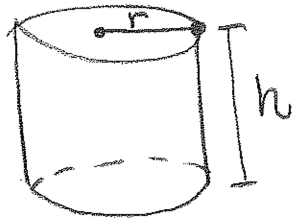
$$\frac{d}{dt}V = \frac{d}{dt}\,25 \cdot 40 \cdot h$$

$$\frac{dV}{dt} = 1000\,\frac{dh}{dt}$$

$$500 = 1000\,\frac{dh}{dt}$$

$$\boxed{\frac{1}{2}\text{ m/min}} = \frac{dh}{dt}$$

(3)



want $\frac{dV}{dr}$

$V = h \cdot \pi r^2$

$\frac{d}{dr} V = \frac{d}{dr} h \pi r^2$

$$\boxed{\frac{dV}{dr} = \frac{dh}{dr} \pi r^2 + 2\pi h r}$$

No values, so this is as far as we can go.

3. Slope Fields

1. very step around $y=0$ K

2. Slope 0 along $x+y=-1$ F

3. No x
neg y ⇒ neg slope
pos y ⇒ pos slope G

4. No y
Slope always pos
Steap slopes H

5. No y
neg, pos slopes
steep around 0 J

6. Symmetric across x, y axes
Slope 1, -1 along $y=x$, $y=-x$ D

7. No x. B
flat around a pos y-value
pos below, neg above

8. No x L
positive slope
almost everywhere

9. No y C
pos everywhere

10. Very flat around $y=0$ I

11. almost always A
neg slope, except for
small y
No x

12. neg in quad I, III E
pos in quad II, IV

4. Exponential Growth and Decay

(1) $y(0) = 100$ $\quad \frac{dy}{dt} = ky$

$y(6) = 450$

$$\int \frac{1}{y}\,dy = \int k\,dt$$

$$\ln y = kt + C$$

$$y = De^{kt}$$

$$100 = De^0 = D \quad \text{so } y = 100e^{kt}$$

$$450 = 100\,e^{k \cdot 6}$$

$$\frac{9}{2} = e^{6k} \iff \frac{\ln\left(\frac{9}{2}\right)}{6} = k$$

$$y = 100e^{\frac{\ln(\frac{9}{2})}{6}t}$$

$$= 100\left(\frac{9}{2}\right)^{t/6}$$

$$y(10) = \boxed{100\left(\frac{9}{2}\right)^{10/6} \approx 1226.56}$$

(2) $y(1) = 200$ $\quad$ Again: $\frac{dy}{dt} = ky$ so

$y(2) = 500$

$$y = Ce^{kt}$$

$$200 = Ce^k \quad \text{and} \quad 500 = Ce^{2k}$$

$$\underbrace{\frac{200}{e^k} = C}_{\text{substitute this}} \qquad 500 = \left(\frac{200}{e^k}\right)e^{2k}$$

$$= \frac{200 \cdot e^k \cdot \cancel{e^k}}{\cancel{e^k}}$$

$$500 = 200 \cdot e^k$$

$$\frac{5}{2} = e^k \iff \ln\left(\frac{5}{2}\right) = k$$

$$\text{so } \frac{200}{e^{\ln(5/2)}} = \frac{200}{5/2} = 80 = C$$

$$y = 80e^{\ln(\frac{5}{2})t} = 80\left(\frac{5}{2}\right)^t$$

So, when $t = 0$, $y(0) = 80\left(\frac{5}{2}\right)^0 = \boxed{80}$

(3) $\frac{dT}{dt} = k(T-30)$

$\int \frac{dT}{T-30} = \int k\,dt$

$\ln(T-30) = kt + C$

$T-30 = De^{kt}$

$T = De^{kt}+30$

$60 = De^{k\cdot 0}+30$

$30 = D$

so $T = 30e^{kt}+30$

$50 = 30e^{k}+30$

$\frac{2}{3} = e^{k}$

$\ln(2/3) = k$

$T = 30e^{\ln(2/3)t}+30$

$T = 30(2/3)^{t}+30$

$40 = 30(2/3)^{t}+30$

$\frac{10}{30} = (2/3)^{t}$

$\ln(1/3) = \ln(2/3)^{t}$

$\ln(1/3) = t\ln(2/3)$

$\frac{\ln(1/3)}{\ln(2/3)} = t$

≈ 2.71 days

(4) $P = Ce^{kt}$

C = initial population
so at t=6, population is 2C
at t=12, population is 4C

$2C = Ce^{k6}$

$2 = e^{6k}$

$\ln 2 = 6k$

$\frac{\ln 2}{6} = k$

When does population equal 100C?

$P = Ce^{\frac{\ln 2}{6}t}$

$= C\cdot 2^{t/6}$

$100C = C\cdot 2^{t/6}$

$100 = 2^{t/6}$

$\ln 100 = t/6 \ln 2$

$\frac{6\ln 100}{\ln 2} = t$

$\approx \boxed{39.86 \text{ hours}}$

5. Euler's Method

1.(a)

x	y	dy/dx	Δy
0	0	$0+0^2$	$0\cdot(.2)=0$
.2	0	$.2+0^2=.2$	$(.2)(.2)=.04$
.4	.04		

$\boxed{y(0.4)\approx .04}$

(b)

x	y	dy/dx	Δx
0	0	0	0
.1	0	$.1+0^2=.1$	$(.1)(.1)=.01$
.2	.01	$.2+.01^2=.2001$	$(.2001)(.1)=.02001$
.3	.03001	$.3+.03001^2=$ $.3009006001$	$(.3009006001)(.1)=$ $.03009006001$
.4	.06010006001		

$y(0.4)\approx .06010006001$

(2) $\frac{dy}{dx} = y + 2x$

x	y	dy/dx	Δy
1	0	2	$(2)(1/2) = 1$
1.5	1	4	$4 \cdot 1/2 = 2$
2	3	$3 + 2 \cdot 2 = 7$	$7 \cdot 1/2 = 3.5$
2.5	6.5	$6.5 + 2 \cdot 2.5 = 11.5$	$11.5 \cdot 1/2 = 5.75$
3	12.25	$12.25 + 2 \cdot 3 = 18.25$	$18.25 \cdot 1/2 = 9.125$
3.5	21.375	$21.375 + 2 \cdot 3.5 = 28.375$	14.1875
4	35.5625		

$$\boxed{y(4) \approx 35.5625}$$

6. Issues in Curve Sketching

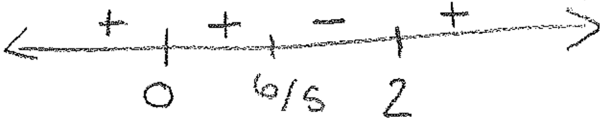
(1) $f(x) = x^5 - 4x^4 + 4x^3$

(a) defined everywhere
(b) cont. everywhere
(c) diff'ble everywhere
(d)
(e) No asymptotes
(f) $f'(x) = 5x^4 - 16x^3 + 12x^2$

$= x^2(5x^2 - 16x + 12)$

$= x^2(5x - 6)(x - 2)$

$f'(x) = 0$ when $x = 0, 6/5, 2$



Increasing: $(-\infty, 6/5] \cup [2, \infty)$
decreasing: $[6/5, 2]$

(h) 6/5 is a local max
2 is a local min

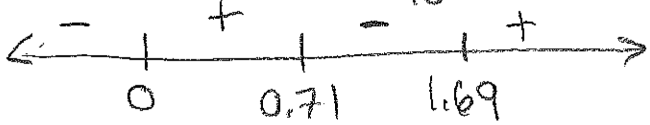
(g) $f''(x) = 20x^3 - 48x^2 + 24x =$

$= 4x(5x^2 - 12x + 6)$

$f''(x) = 0$ when $x = 0$

$x = \frac{12 \pm \sqrt{244 - 4 \cdot 5 \cdot 6}}{10}$

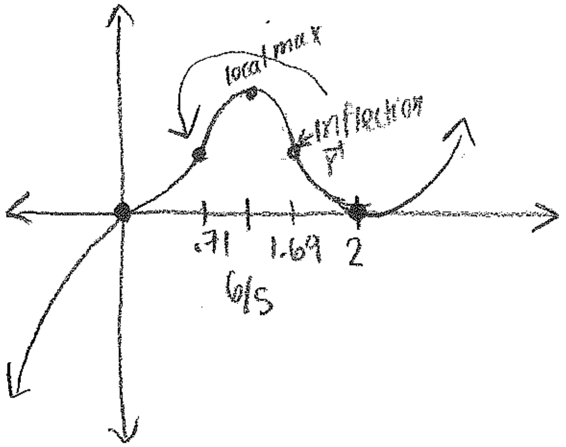
$= \frac{12 \pm \sqrt{24}}{10} = 1.69$ OR 0.71



concave up: $(0, .71) \cup (1.69, \infty)$
down: $(-\infty, 0) \cup (.71, 1.69)$

(i) Inflection pts:
$x = 0, .71, 1.69$

Sketch



(2) $f(x) = \frac{x^3}{x^2+1}$

Domain: $(-\infty,\infty)$

Cont everywhere

diff'ble everywhere

$$f'(x) = \frac{3x^2\cdot(x^2+1) - 2x\cdot x^3}{(x^2+1)^2} = \frac{3x^4+3x^2-2x^4}{(x^2+1)^2} = \frac{x^4+3x^2}{(x^2+1)^2} = \underbrace{\frac{x^2(x^2+3)}{(x^2+1)^2}}_{f'(x)}$$

$$f''(x) = \frac{(4x^3+6x)(x^2+1)^2 - 2(x^2+1)\cdot 2x\cdot(x^4+3x^2)}{(x^2+1)^4}$$

$$= \frac{(4x^3+6x)(x^4+2x^2+1) - (4x^3+4x)(x^4+3x^2)}{(x^2+1)^4}$$

$$= \frac{4x^7+8x^5+4x^3+6x^5+12x^3+6x-4x^7-12x^5-4x^5-12x^3}{(x^2+1)^4}$$

$$= -\frac{2x^5+4x^3+6x}{(x^2+1)^4} = \frac{-2x(-x^4+2x^2+3)}{(x^2+1)^4}$$

$$= \underbrace{\frac{2x(-x^2+2)(x^2+1)}{(x^2+1)^4}}_{f''(x)}$$

horizontal assymps:

$\lim_{x\to\infty} \frac{x^3}{x^2+1} = \infty$

$\lim_{x\to-\infty} \frac{x^3}{x^2+1} = -\infty$

No assymptotes

(f), (h)

$f'(x) = \frac{x^2(x^2+3)}{(x^2+1)^2}$

$f'(x) = 0$ when $x = 0$

+ + on either side of 0

increasing everywhere

No local max/min

(g), (i)

$f''(x) = \frac{2x(-x^2+2)(x^2+1)}{(x^2+1)^4}$

$f''(x) = 0$ when

$x = 0, \pm\sqrt{2}$

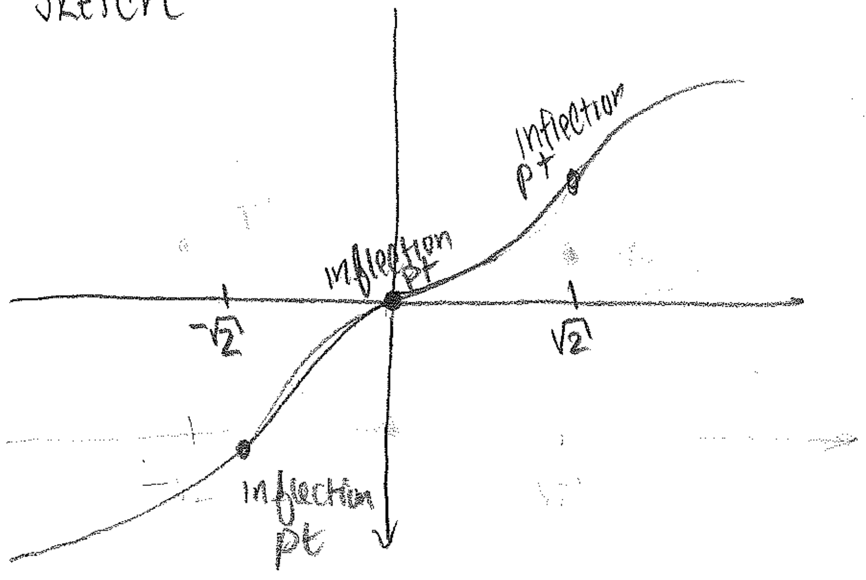
+ | − | + | − at $-\sqrt{2}$, 0, $\sqrt{2}$

inflection points at $-\sqrt{2}, \sqrt{2}, 0$

Concave up: $(-\infty,-\sqrt{2}) \cup (0,\sqrt{2})$

Concave down: $(-\sqrt{2}, 0 \cup(\sqrt{2},\infty)$

Sketch



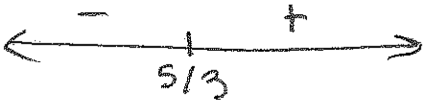
(3) $f(x) = (x-2)^2(x-1) = (x^2-4x+4)(x-1) = x^3-4x^2+4x-x^2+4x-4$
$= x^3-5x^2+8x-4$

$f'(x) = 3x^2-10x+8 = (3x-4)(x-2)$
$f''(x) = 6x-10 = 2(3x-5)$

- f is defined, cont, diff'ble everywhere
- No assymptotes
- Inc/Dec: $f'(x)=0$ when $x = 4/3, 2$

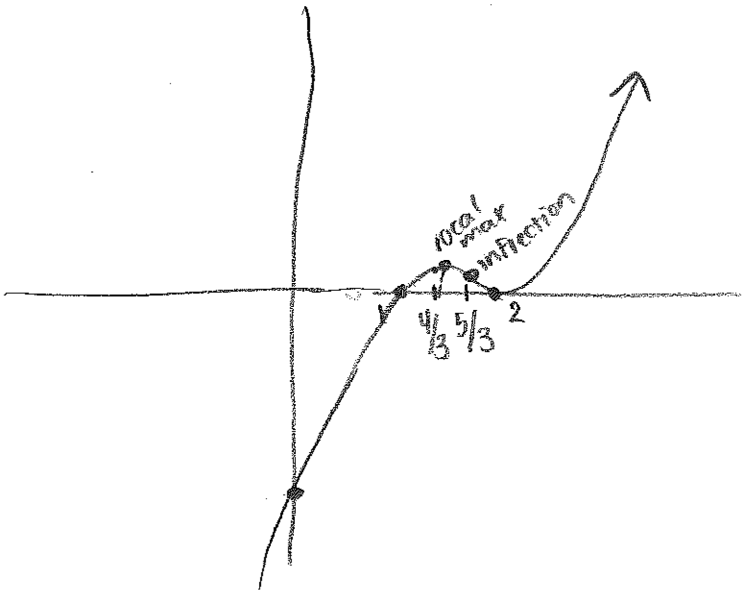
increasing on: $(-\infty, 4/3] \cup [2, \infty)$
decreasing on: $[4/3, 2]$
local max @ 4/3
local min @ 2

- Concavity: $f''(x)=0$ when $x = 5/3$



concave up: $[5/3, \infty)$
concave down: $(-\infty, 5/3]$
inflection pt @ 5/3

- $f(x)=0$ @ 1, 2
- $f(0) = -4$



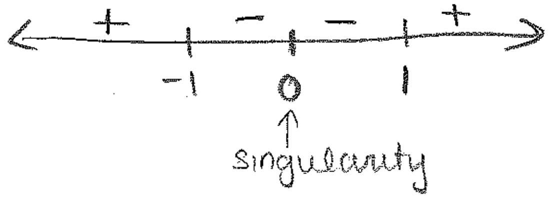
(4) $f(x) = x + 1/x$ $f'(x) = 1 - 1/x^2$ $f''(x) = \frac{2}{x^3}$

Domain: $(-\infty, 0) \cup (0, \infty)$
Diff'ble: $(-\infty, 0) \cup (0, \infty)$

Assymptotes: vertical @ $x=0$; $\lim_{x \to 0^-} x + \frac{1}{x} = -\infty$

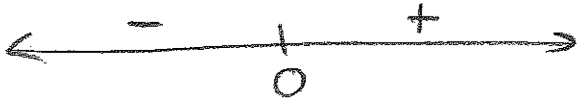
$\lim_{x \to 0^+} x + \frac{1}{x} = \infty$

Inc/dec: $f'(x) = 0$ when $x = \pm 1$



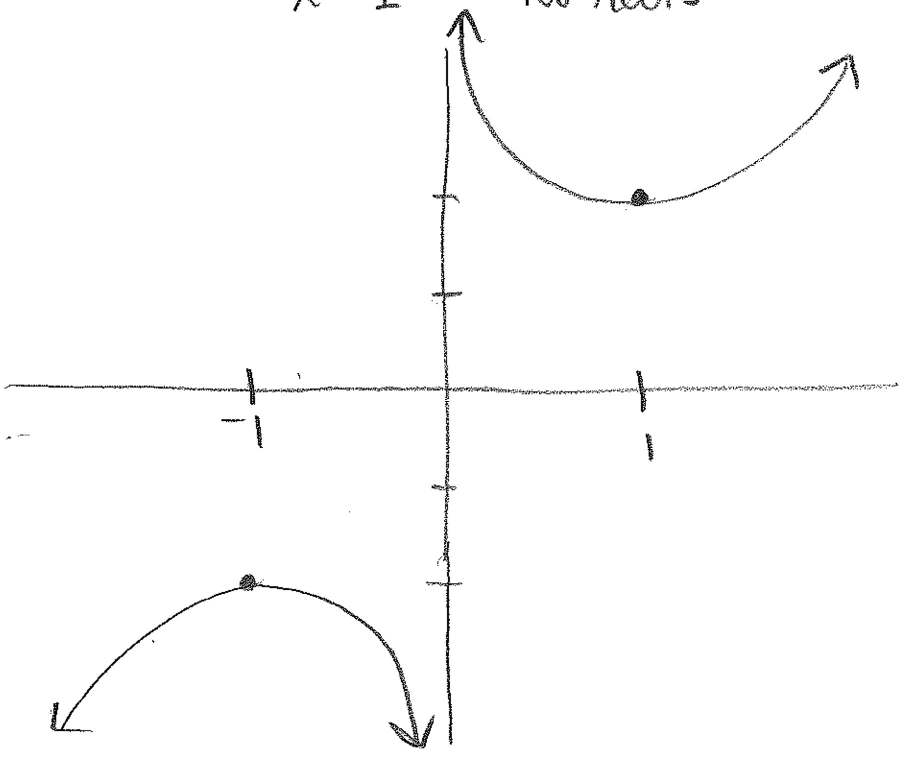
dec: $[-1, 0) \cup (0, 1]$
inc: $(-\infty, -1] \cup [1, \infty)$
local max @ −1
" min @ 1

Concavity: $f''(x) = 0$ never
undefined at $x = 0$



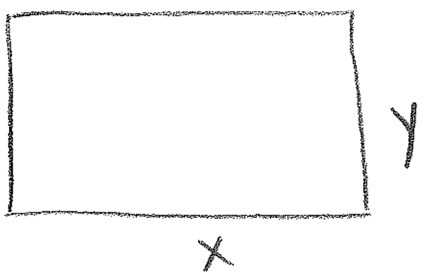
Concave up: $(0, \infty)$
" down: $(-\infty, 0)$
No inflection pts.

Roots: $0 = x + 1/x$
$-x = 1/x$ Never possible
$-x^2 = 1$ No roots



7. Optimization

(1)



$96 = xy \iff y = 96/x$

$\underbrace{P}_{\text{minimize this}} = 2x+2y$

$0 \le x <$

$P = 2x + 2\left(\frac{96}{x}\right)$

$\frac{dP}{dx} = 2 - 192 \cdot \frac{1}{x^2}$

$2 - 192 \cdot \frac{1}{x^2} = 0$

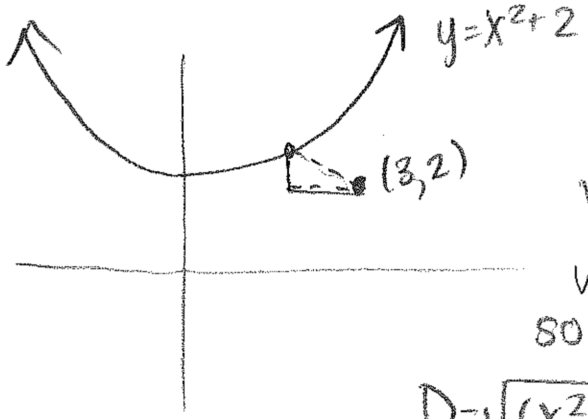
$x^2 = 96$

$x = \pm\sqrt{96}$ (only pos makes sense)

minimum perimeter:

$2\sqrt{96} + \frac{192}{\sqrt{96}} \approx \boxed{39.19 \text{ cm}}$

(2)



distance is

$\sqrt{(y-2)^2 + (x-3)^2}$ by Pythagoras

we know $y = x^2+2$

so we get

$D = \sqrt{(x^2+2-2)^2 + (x-3)^2}$

$\underbrace{D}_{\text{minimize}} = \sqrt{x^4+x^2-6x+9}$

$\frac{dD}{dx} = \underbrace{\frac{1}{2\sqrt{x^4+x^2-6x+9}} \cdot (4x^3+2x-6)}$

$= 0$ when $\underline{4x^3-2x-6} = 0$

$2x^3+x-3=0$

$x=1$ is a solution

so $2x^3+x-3 = (x-1)\underbrace{(2x^2+2x+3)}_{\text{no real sol'ts}}$

− 1 +

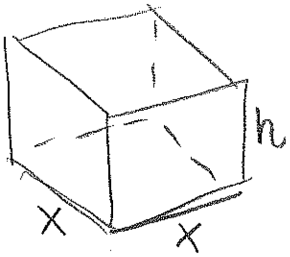
dec ↑ inc

minimum

when $x=1$, minimum distance.

so when jet is at (1,3), minimum distance

(3)



$9 = x^2 + 4xh$

$\frac{9-x^2}{4x} = h$

$V = x^2h$

$V = x^2\left(\frac{9-x^2}{4x}\right)$

$= \frac{9x}{4} - \frac{x^3}{4}$

$\frac{dV}{dx} = \underbrace{\frac{9}{4} - \frac{3}{4}x^2}$

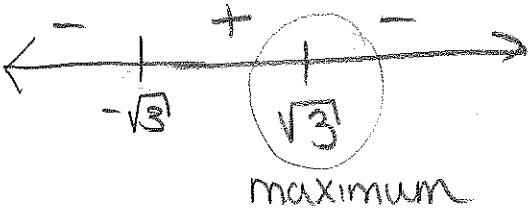
$\frac{9}{4} - \frac{3}{4}x^2 = 0$

$\frac{9}{4} = \frac{3}{4}x^2$

$3 = x^2$

$\sqrt{3} = \pm x$

$x = \sqrt{3}$



Max volume:

$V = \sqrt{3}^2\left(\frac{9-\sqrt{3}^2}{4\sqrt{3}}\right)$

$= \frac{18}{4\sqrt{3}} \approx 2.60 \text{ cm}^3$