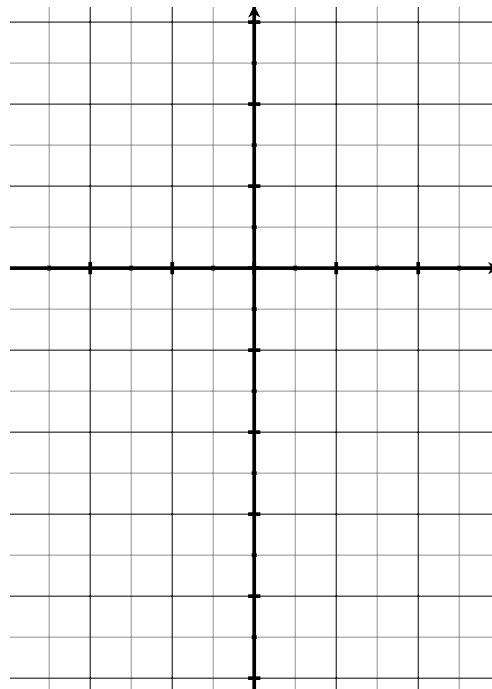


## Slope Fields and Euler's Method

### Warm up

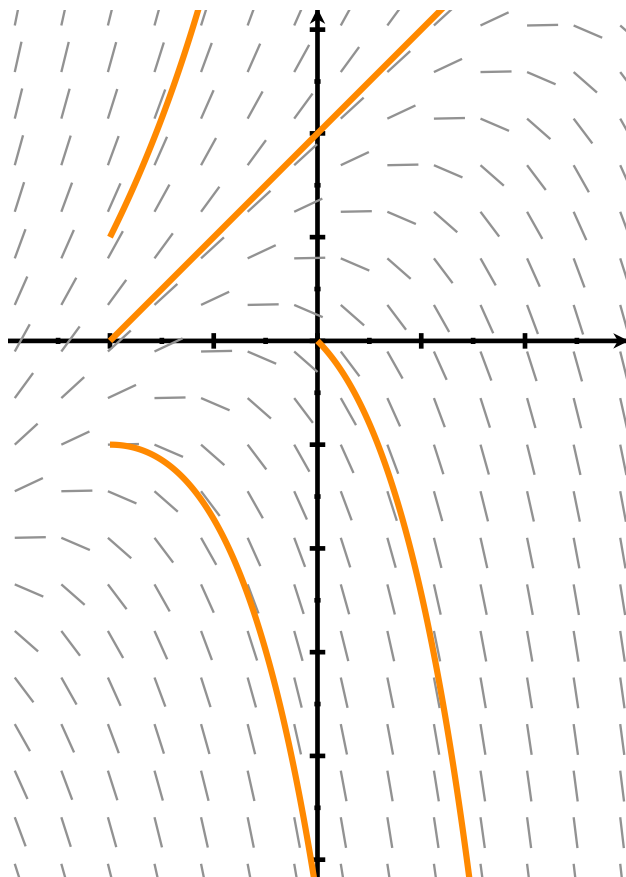
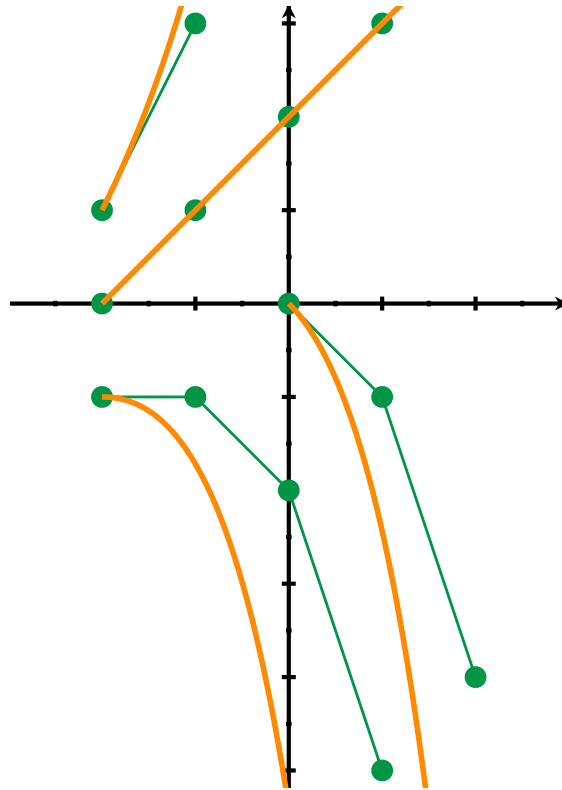
Suppose  $\frac{dy}{dx} = y - x - 1$ . Sketch part of the slope field for the following points:

$x$	$y$	$\frac{dy}{dx}$
-2	0	
-2	1	
-2	-1	
-1	1	
-1	-1	
0	2	
0	0	
0	-2	
1	-1	



Suppose  $\frac{dy}{dx} = y - x - 1$ . Sketch part of the slope field for the following points:

$x$	$y$	$\frac{dy}{dx}$
-2	0	1
-2	1	2
-2	-1	0
-1	1	1
-1	-1	-1
0	2	1
0	0	-1
0	-2	-3
1	-1	-3



## Euler's Method

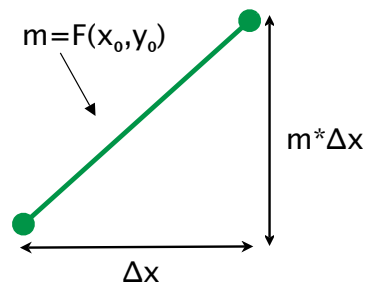
Assume that you have an IVP that looks like

$$\frac{dy}{dx} = F(x, y), \quad y(x_0) = y_0$$

Pick an increment of  $x$ -steps, called  $\Delta x$ .

Start at  $(x_0, y_0)$ , and plot a segment with run  $\Delta x$  and slope  $F(x_0, y_0)$ . The end is  $(x_1, y_1)$ . From each  $(x_i, y_i)$ , generate  $(x_{i+1}, y_{i+1})$  by plotting a segment with run  $\Delta x$  and slope  $F(x_i, y_i)$ .

$$\begin{aligned} x_{i+1} &= x_i + \Delta x \\ y_{i+1} &= y_i + \Delta x * F(x_i, y_i) \end{aligned}$$

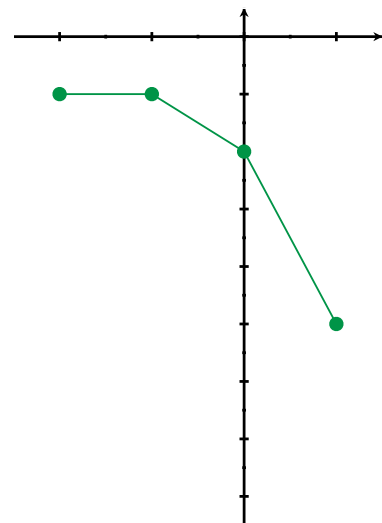


Back to  $\frac{dy}{dx} = y - x - 1 = F(x, y), \quad P_0 = (-2, -1)$ .

$$\begin{aligned} x_{i+1} &= x_i + \Delta x \\ y_{i+1} &= y_i + \Delta x * (y_i - x_i - 1) \end{aligned}$$

Example:  $\Delta x = 1$

$i$	$x_i$	$y_i$	$F(x_i, y_i)$	$y_{i+1}$
0	-2	-1	0	$-1+1*0$
1	-1	-1	-1	$-1+1*(-1)$
2	0	-2	-3	$-2+1*(-3)$
3	1	-5		



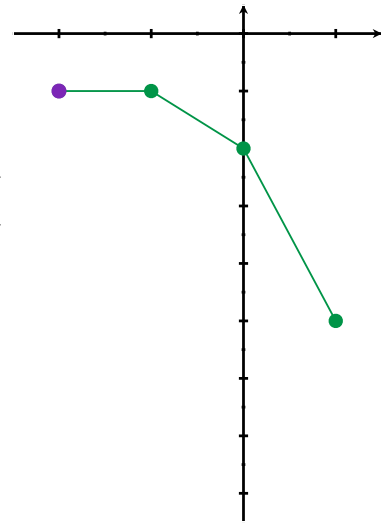
$$\Delta x = 1, y_3 = -5$$

Back to  $\frac{dy}{dx} = y - x - 1 = F(x, y)$ ,  $P_0 = (-2, -1)$ .

$$x_{i+1} = x_i + \Delta x$$
$$y_{i+1} = y_i + \Delta x * (y_i - x_i - 1)$$

Example:  $\Delta x = 1/2$

$i$	$x_i$	$y_i$	$F(x_i, y_i)$	$y_{i+1}$
0	-2	-1		
1				
2				
3				



$\Delta x = 1, y_3 = -5$

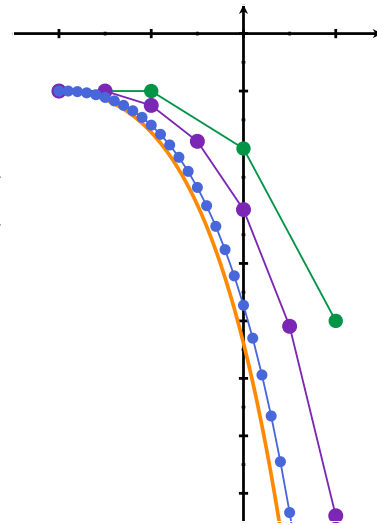
Back to  $\frac{dy}{dx} = y - x - 1 = F(x, y)$ ,  $P_0 = (-2, -1)$ .

$$x_{i+1} = x_i + \Delta x$$

$$y_{i+1} = y_i + \Delta x * (y_i - x_i - 1)$$

Example:  $\Delta x = 1/2$

$i$	$x_i$	$y_i$	$F(x_i, y_i)$	$y_{i+1}$
0	-2	-1	0	$-1 + \frac{1}{2} * 0$
1	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	$-1 + \frac{1}{2} * (-\frac{1}{2})$
2	-1	$-\frac{5}{4}$	$-\frac{5}{4}$	$-\frac{5}{4} + \frac{1}{2} * (-\frac{5}{4})$
3	$-\frac{1}{2}$	$-\frac{15}{8}$	$-\frac{19}{8}$	$-\frac{15}{8} + \frac{1}{2} * (-\frac{19}{8})$
4	0	-3.0625	-4.0625	
5	$\frac{1}{2}$	-5.0938	-6.5938	
6	1	-8.3906 = $y_6$		



$\Delta x = 1, y_3 = -5$

$\Delta x = .1, y_{30} = -14.449$

Actual solution:  $y = -e^{x+2} + x + 2$ ,  $y(1) \approx -17.0855$

## Another example

If  $\frac{dy}{dx} = \frac{y^2 + 3y}{x + 4}$  and  $y(0) = -1$ , what is  $y(3)$ ?

**Estimate:** Try  $\Delta x = 1$

$$x_0 = 1, y_0 = -1$$

$$m_0 = \frac{(-1)^2 + 3(-1)}{0+4} = -0.5$$

$$x_1 = 0 + 1 = 1 \quad y_1 = -1 + 1 * (-0.5) = -1.5$$

$$m_1 =$$

$$x_2 =$$

$$y_2 =$$

$$m_2 =$$

$$x_3 =$$

$$y_3 =$$

Spreadsheet set up:

in cell...	A1	B1	C1	D1	A2	B2	C2	F2
put...	i	$x_i$	$y_i$	$m_i$	0	$x_0$	$y_0$	$\Delta x$

(In the first example,  $x_0$  was -2 and  $y_0$  was -1 and  $\Delta x$  was 1,  $\frac{1}{2}, \dots$ )

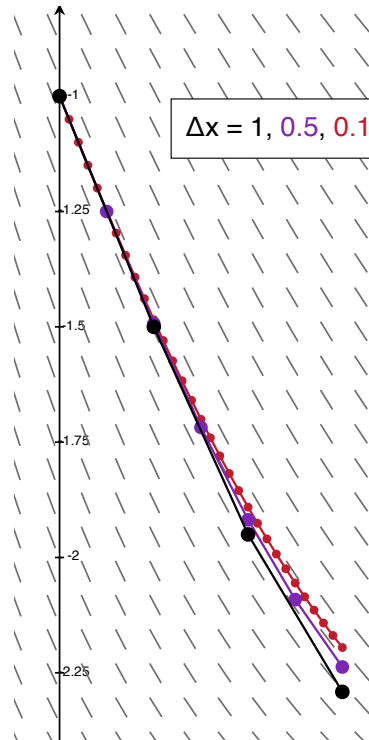
in cell...	D2	B3	C3
put...	$=F(B2, C2)$	$=B2+\$F\$2$	$=C2+\$F\$2*D2$
		$x_{i+1} = x_i + \Delta x$	$y_{i+1} = y_i + \Delta x * m_i$

(In the last example,  $F(B2, C2)$  was  $C2 - B2 - 1$ )

i	$x_i$	$y_i$	$m_i$	Dx
0	0	-1	-0.5	1
1	1	-1.5	-0.45	
2	2	-1.95	-0.34125	
3	3	-2.29125		

i	$x_i$	$y_i$	$m_i$	Dx 0.5
0	0	-1	-0.5	
1	0.5	-1.25	-0.4861111111	
2	1	-1.493055556	-0.449990355	
3	1.5	-1.718050733	-0.40044616	
4	2	-1.918273813	-0.34584117	
5	2.5	-2.091194398	-0.292382951	
6	3	-2.237385873		

i	$x_i$	$y_i$	$m_i$	Dx 0.1
0	0	-1	-0.5	
1	0.1	-1.05	-0.499390244	
2	0.2	-1.099939024	-0.497607432	
3	0.3	-1.149699768	-0.494718546	
4	0.4	-1.199171622	-0.490795974	
5	0.5	-1.24825122	-0.485916123	
6	0.6	-1.296842832	-0.480158079	
7	0.7	-1.34485864	-0.473602374	
8	0.8	-1.392218877	-0.466329839	
9	0.9	-1.438851861	-0.458420593	
10	1	-1.48469392	-0.449953145	
11	1.1	-1.529689235	-0.441003637	
12	1.2	-1.573789599	-0.431645211	
13	1.3	-1.61695412	-0.421947497	
14	1.4	-1.659148869	-0.411976229	
15	1.5	-1.700346492	-0.401792961	
16	1.6	-1.740525788	-0.391454883	
17	1.7	-1.779671277	-0.381014733	
18	1.8	-1.81777275	-0.370520772	
19	1.9	-1.854824827	-0.360016838	
20	2	-1.890826511	-0.34954244	
21	2.1	-1.925780755	-0.33913291	
22	2.2	-1.959694046	-0.328819578	
23	2.3	-1.992576004	-0.318629981	
24	2.4	-2.024439002	-0.308588083	
25	2.5	-2.05529781	-0.298714514	
26	2.6	-2.085169262	-0.289026808	
27	2.7	-2.114071942	-0.279539649	
28	2.8	-2.142025907	-0.270265108	
29	2.9	-2.169052418	-0.261212879	
30	3	-2.195173706		





Back to "If  $\frac{dy}{dx} = \frac{y^2 + 3y}{x + 4}$  and  $y(0) = -1$ , what is  $y(3)$ ?"

Can I just solve? Separate...

$$\int \frac{1}{y^2 + 3y} dy = \int \frac{1}{x + 4} dx = \ln|x + 4| + C$$

$$\frac{1}{3} \int \frac{1}{y} dy - \frac{1}{3} \int \frac{1}{y + 3} dy = \text{check LHS: } \frac{1}{3} \left( \frac{1}{y} - \frac{1}{y + 3} \right) = \frac{1}{3} \left( \frac{y + 3 - y}{y(y + 3)} \right)$$

$$\frac{1}{3} \ln|y| - \frac{1}{3} \ln|y + 3| = \frac{1}{3} \ln \left| \frac{y}{y + 3} \right| = \frac{1}{3} \frac{3}{y^2 + 3y} \checkmark$$

Solve for y:

$$\frac{1}{3} \ln \left| \frac{y}{y + 3} \right| = \ln|x + 4| + C \quad \text{so} \quad \ln \left| \frac{y}{y + 3} \right| = 3 \ln|x + 4| + 3C$$

Plug in point:

$$y(0) = -1 \quad \text{means} \quad \ln \left| \frac{-1}{-1 + 3} \right| = 3 \ln|0 + 4| + 3C$$

$$\text{so } 3C = \ln(1/2) - 3 \ln(4) = \ln \left( \frac{2^{-1}}{(2^2)^3} \right) = \ln(2^{-1-6}) = \ln(2^{-7})$$

(computation one does in the privacy of one's home)

Back to "If  $\frac{dy}{dx} = \frac{y^2 + 3y}{x + 4}$  and  $y(0) = -1$ , what is  $y(3)$ ?"

Solve for  $y$ :

$$\frac{1}{3} \ln \left| \frac{y}{y+3} \right| = \ln|x+4| + C \quad \text{so} \quad \ln \left| \frac{y}{y+3} \right| = 3 \ln|x+4| + 3C$$

$$= 3 \ln|x+4| + \ln(2^{-7})$$

So

$$\left| \frac{y}{y+3} \right| = e^{3 \ln|x+4| + \ln(2^{-7})} = e^{3 \ln|x+4|} e^{\ln(2^{-7})} = \left( e^{\ln|x+4|} \right)^3 2^{-7}$$

$$= |x+4|^3 2^{-7}. \quad \text{So} \quad \frac{y}{y+3} = -2^{-7}|x+4|^3 \quad (- \text{ b/c } y(0) = -1)$$

Therefore, if  $A(x) = 2^{-7}|x+4|^3$ , multiplying both sides by  $y+3$  gives:

$$y = -A(x)(y+3) = -A(x) * y - A(x) * 3,$$

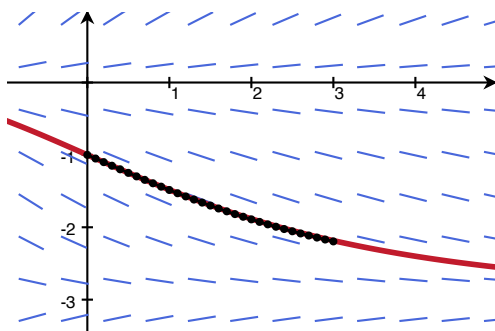
And so

$$-3A(x) = y + A(x) * y = (1 + A(x))y$$

So, finally, divide by  $(1 + A(x))$  and sub back:

$$y = \frac{-3 * A(x)}{1 + A(x)} = \frac{-3 * 2^{-7}|x+4|^3}{1 + 2^{-7}|x+4|^3}$$

Compare  $y = \frac{-3 * 2^{-7}|x+4|^3}{1 + 2^{-7}|x+4|^3}$  to our estimate:



$$y(3) = \frac{-3 * 2^{-7} 7^3}{1 + 2^{-7} 7^3} = -2.18471 \dots$$

$$\text{Euler w } \Delta x = .1 : y(3) \approx -2.19517 \dots$$

Notice that  $\lim_{x \rightarrow \infty} \frac{-3 * 2^{-7}|x+4|^3}{1 + 2^{-7}|x+4|^3} = \lim_{x \rightarrow \infty} \frac{-3 * 2^{-7} x^3}{2^{-7} x^3} = -3$

