Area between curves

Putting FTC and u-substitution together

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Step 2: Use your answer to compute

$$\int_0^{\sqrt{\pi/2}} x \sin(x^2 + 3)\, dx = F(\pi/2) - F(0).$$

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Let $u = x^2$.

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Let $u = x^2$. So $du = 2x\ dx$, and $\frac{1}{2}\ du = x\ dx$.

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Therefore

$$\int x\sin(x^2)\ dx = \int \sin(u) * \frac{1}{2}\ du$$

$$= -\frac{1}{2}\cos(u) + C = -\frac{1}{2}\cos(x^2) + C$$

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$$\int_0^{\sqrt{\pi/2}} x\sin(x^2+3)\ dx = F(\pi/2) - F(0).$$

$$\int_0^{\pi/2} x\sin(x^2+3)\ dx = -\frac{1}{2}\cos((\sqrt{\pi/2})^2) - \left(-\frac{1}{2}\cos(0^2)\right) = 1/2$$

Warm-up

1. Calculate the area under the curve $y = -x^2 + 5x - 6$ between $x = 1$ and $x = 2$.
2. Calculate the area contained between the curve $y = -x^2 + 5x - 6$ and the x-axis.
   (Draw a picture. Where does $y = -x^2 + 5x - 6$ intersect the x-axis? Those are your bounds.)
3. Calculate the area contained between the curve $y = x^2 - 5x + 6$ and the x-axis.
   (Draw a picture. Your answer should be positive — we want *area*.)

Areas Between Curves

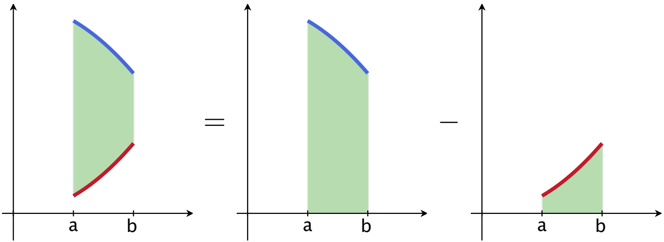
We know that if f is a continuous nonnegative function on the interval $[a, b]$, then $\int_a^b f(x)dx$ is the area under the graph of f and above the interval.

Now suppose we are given two continuous functions, $f(x)$ and $g(x)$ so that $g(x) \leq f(x)$ for all x in the interval $[a, b]$.

How do we find the area bounded by the two functions over that interval?

f = top function

g = bottom function



$$\text{Area between } f \text{ and } g = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b f(x) - g(x)dx$$

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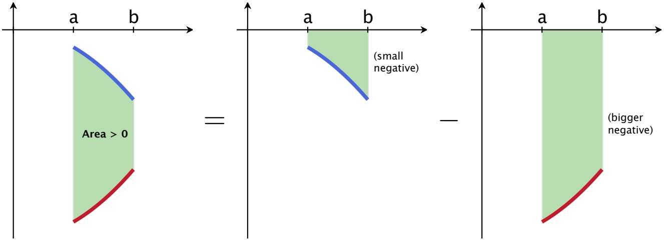

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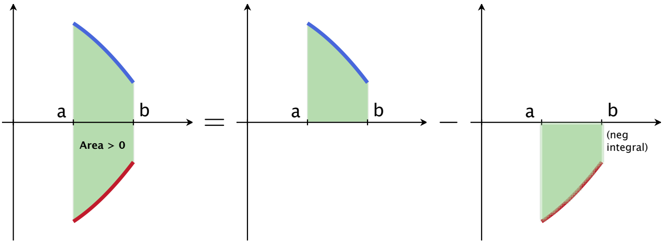

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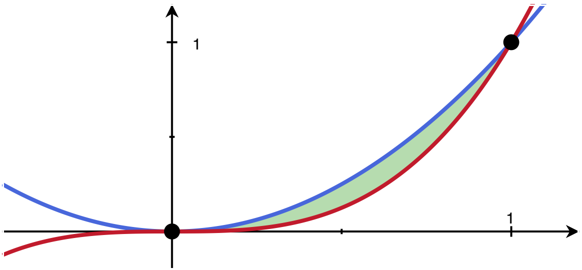
Example

Find the area of the region between the graphs of $y = x^2$ and $y = x^3$ for $0 \leq x \leq 1$.



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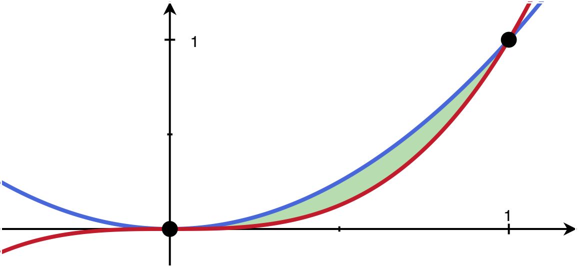

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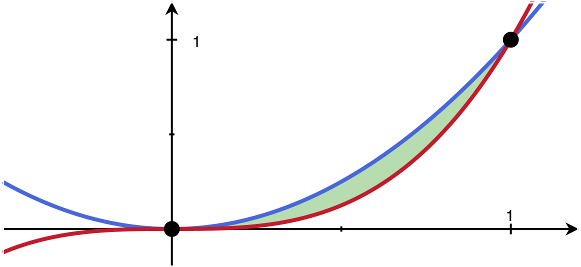

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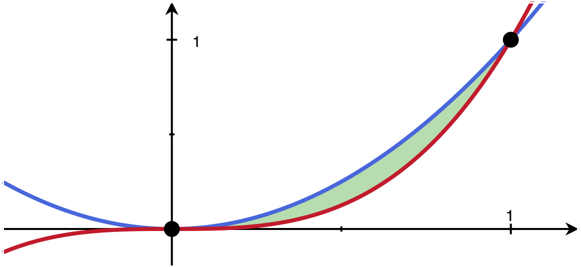

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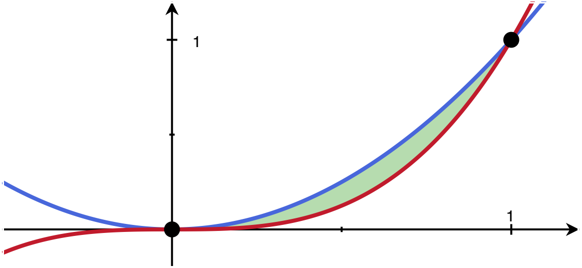

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So $$\text{Area} = \int_0^1 x^2 - x^3 dx$$

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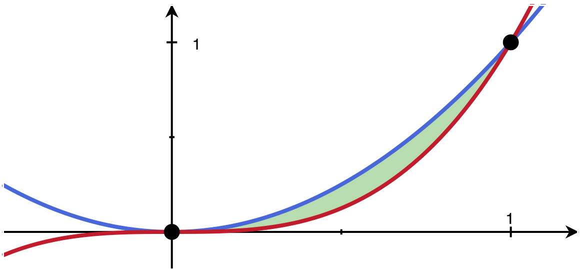

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So Area $= \displaystyle\int_0^1 x^2 - x^3 dx = \frac{1}{3}x^3 - \frac{1}{4}x^4 \Big|_{x=0}^{1} = \boxed{\left(\tfrac{1}{3} - \tfrac{1}{4}\right) - 0} > 0 \checkmark$

Example

Find the area of the region bounded by the two curves $y = x^3 - 9x$ and $y = 9 - x^2$.

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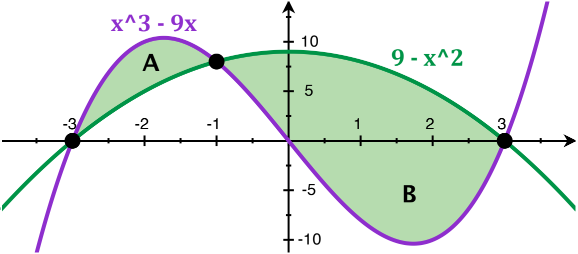

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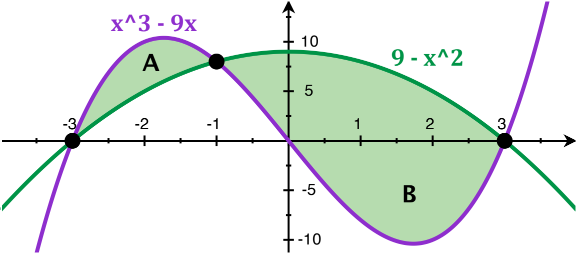

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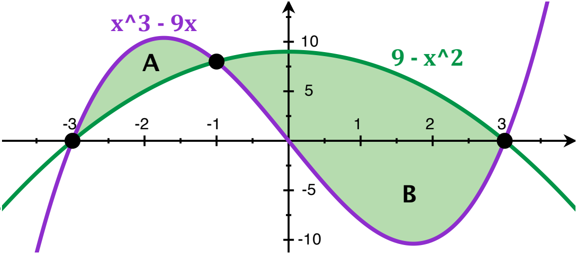

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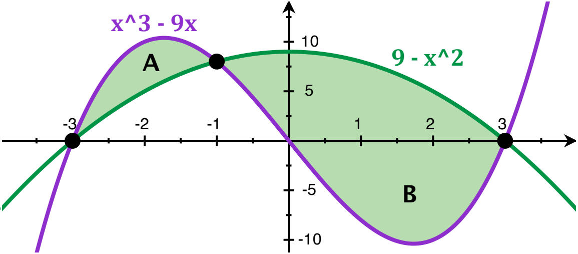

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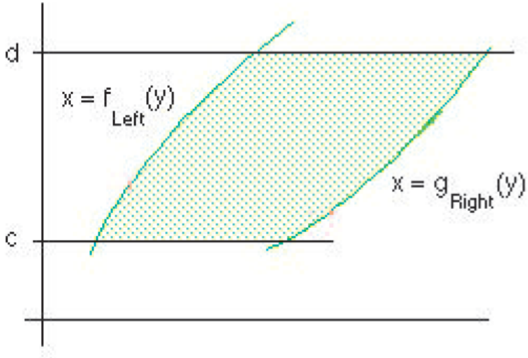
$$\text{Area B} = \int_{-1}^{3} (9 - x^2) - (x^3 - 9x) dx = -\int_{-1}^{3} x^3 + x^2 - 9x - 9 \, dx$$

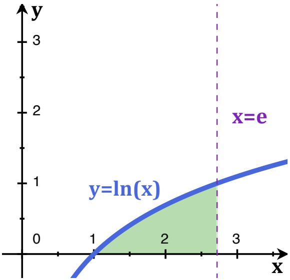
Functions of y

We could just as well consider two functions of y, say, $x = f_{Left}(y)$ and $x = g_{Right}(y)$ defined on the interval $[c, d]$.



Area Between the Two Curves

Find the area under the graph of $y = \ln x$ and above the interval $[1, e]$ on the x-axis.



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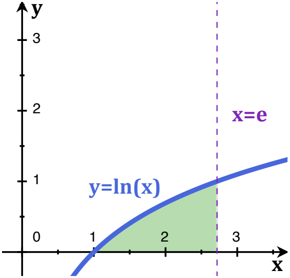

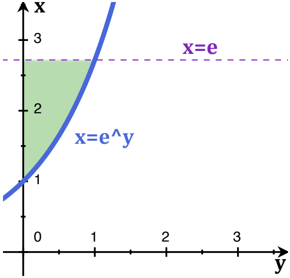

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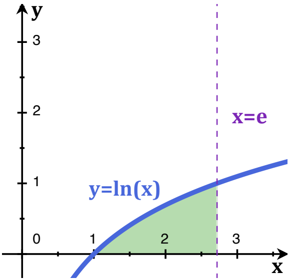

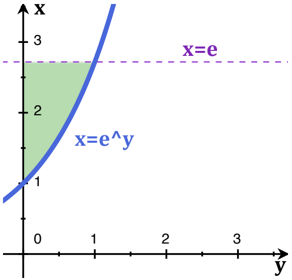

$$\text{area} = \int_{y=0}^{1} e - e^{y} \; dy$$

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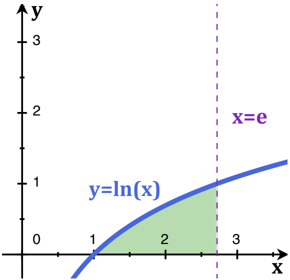

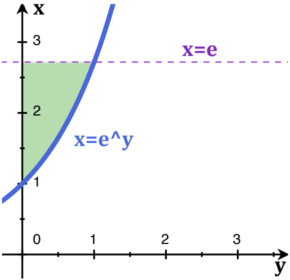

$$\text{area} = \int_{y=0}^{1} e - e^y \; dy = (e * y - e^y)\big|_{y=0}^{1} = (e - e) - (0 - 1) = 1.$$