Area between curves

Putting FTC and u-substitution together

Q. Calculate $\int_0^{\sqrt{\pi/2}} x \sin(x^2)\ dx$.

A. Separate your solution into two steps.

Step 1: Find the antiderivative $F(x)$ of $f(x) = x\sin(x^2)$.

Let $u = x^2$. So $du = 2x\ dx$, and $\frac{1}{2}\ du = x\ dx$.
Therefore

$$\int x\sin(x^2)\ dx = \int \sin(u) * \frac{1}{2}\ du$$
$$= -\frac{1}{2}\cos(u) + C = -\frac{1}{2}\cos(x^2) + C$$

Step 2: Use your answer to compute

$$\int_0^{\sqrt{\pi/2}} x\sin(x^2+3)\ dx = F(\pi/2) - F(0).$$

$$\int_0^{\pi/2} x\sin(x^2+3)\ dx = -\frac{1}{2}\cos((\sqrt{\pi/2})^2) - \left(-\frac{1}{2}\cos(0^2)\right) = 1/2$$

Warm-up

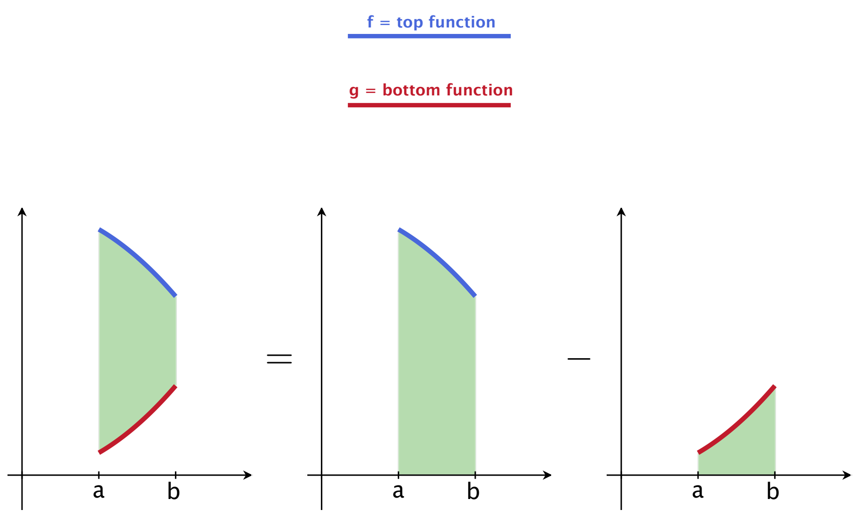
1. Calculate the area under the curve $y = -x^2 + 5x - 6$ between $x = 1$ and $x = 2$.
2. Calculate the area contained between the curve $y = -x^2 + 5x - 6$ and the x-axis.
   (Draw a picture. Where does $y = -x^2 + 5x - 6$ intersect the x-axis? Those are your bounds.)
3. Calculate the area contained between the curve $y = x^2 - 5x + 6$ and the x-axis.
   (Draw a picture. Your answer should be positive — we want *area*.)

Areas Between Curves

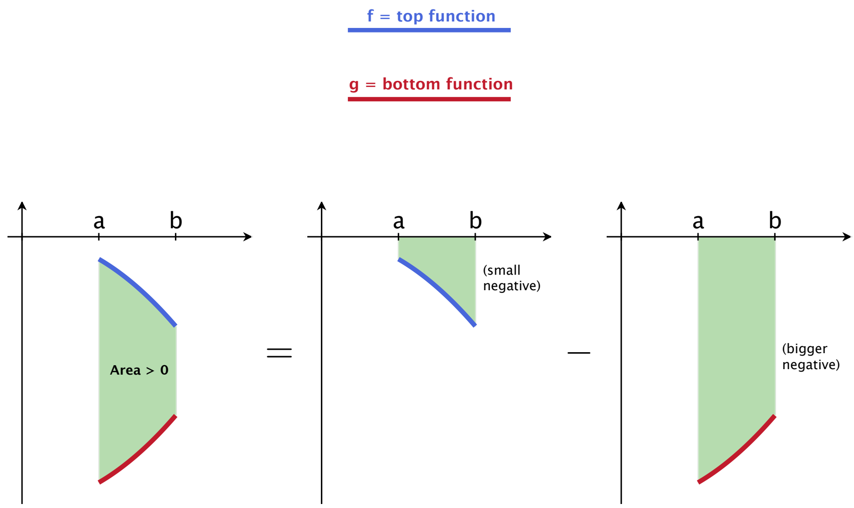
We know that if f is a continuous nonnegative function on the interval $[a, b]$, then $\int_a^b f(x)dx$ is the area under the graph of f and above the interval.

Now suppose we are given two continuous functions, $f(x)$ and $g(x)$ so that $g(x) \leq f(x)$ for all x in the interval $[a, b]$.

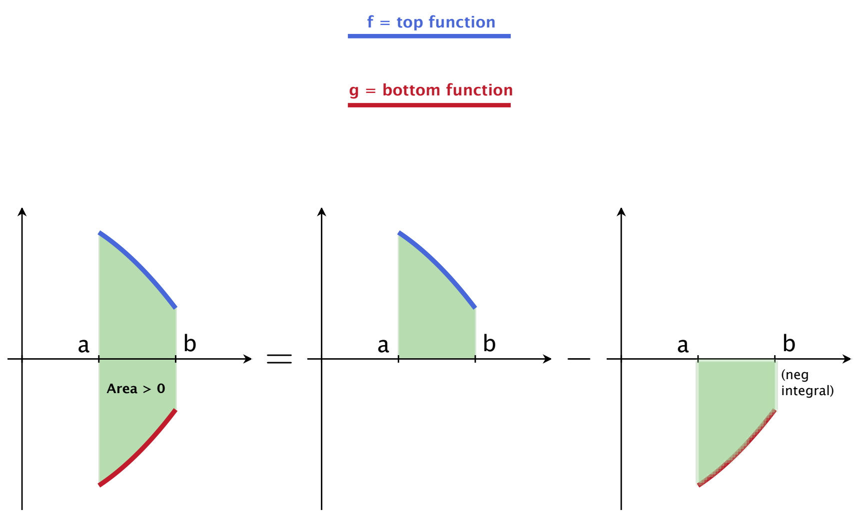
How do we find the area bounded by the two functions over that interval?



Area between f and $g = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b f(x) - g(x)dx$



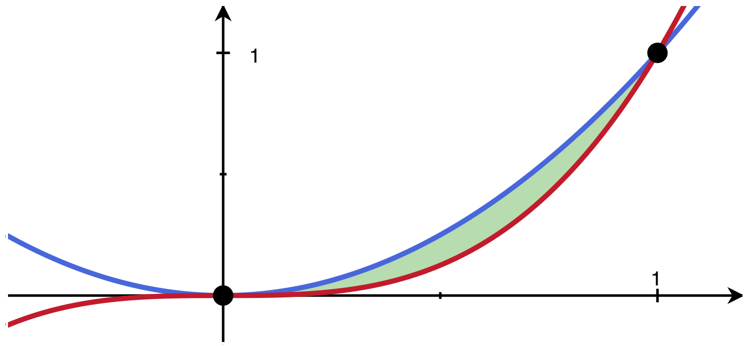
$$\text{Area between } f \text{ and } g = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b f(x) - g(x)dx$$



$$\text{Area between } f \text{ and } g = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b f(x) - g(x)dx$$

Example

Find the area of the region between the graphs of $y = x^2$ and $y = x^3$ for $0 \leq x \leq 1$.



Top: x^2 Bottom: x^3

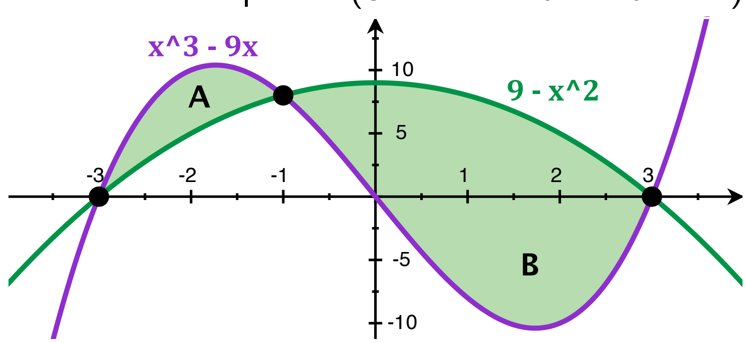
Intersections: where does $x^2 = x^3$? $\boxed{x = 0 \text{ or } 1}$

So $$\text{Area} = \int_0^1 x^2 - x^3 dx = \frac{1}{3}x^3 - \frac{1}{4}x^4\Big|_{x=0}^{1} = \boxed{\left(\tfrac{1}{3} - \tfrac{1}{4}\right) - 0} > 0\checkmark$$

Example

Find the area of the region bounded by the two curves $y = x^3 - 9x$ and $y = 9 - x^2$.

1. Check for intersection points (Solve $x^3 - 9x = 9 - x^2$).



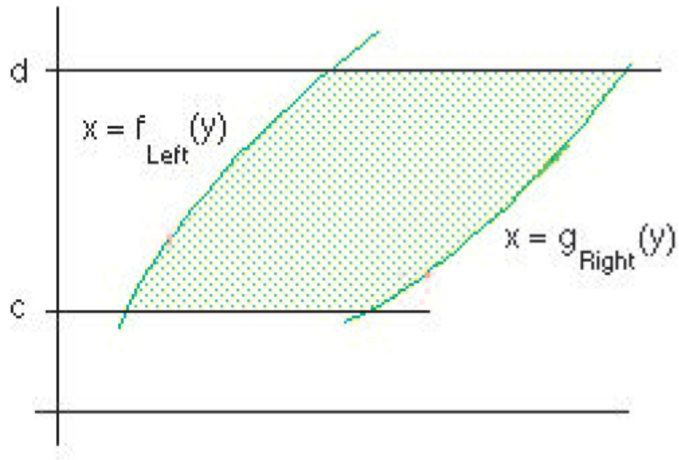
2. Area = Area A + Area B

$$\text{Area A} = \int_{-3}^{-1} (x^3 - 9x) - (9 - x^2) dx = \int_{-3}^{-1} x^3 + x^2 - 9x - 9\, dx$$

$$\text{Area B} = \int_{-1}^{3} (9 - x^2) - (x^3 - 9x) dx = -\int_{-1}^{3} x^3 + x^2 - 9x - 9\, dx$$

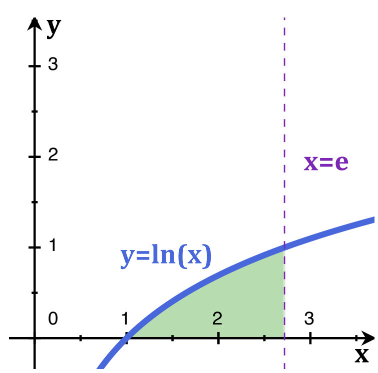
Functions of y

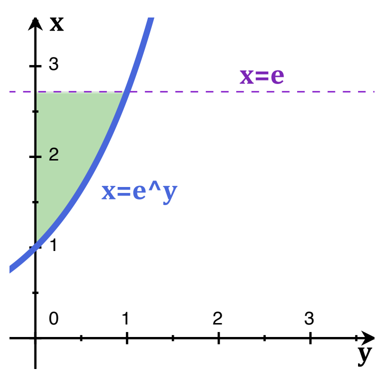
We could just as well consider two functions of y, say, $x = f_{Left}(y)$ and $x = g_{Right}(y)$ defined on the interval $[c, d]$.



Area Between the Two Curves

Find the area under the graph of $y = \ln x$ and above the interval $[1, e]$ on the x-axis.

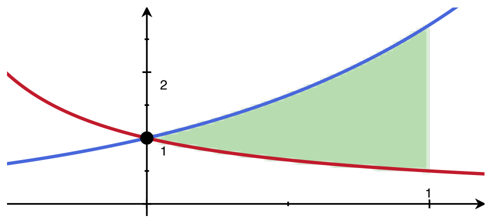




$$\text{area} = \int_{y=0}^{1} e - e^y \; dy = (e * y - e^y)|_{y=0}^{1} = (e - e) - (0 - 1) = 1.$$

Worksheet: Area between curves

Example 1:
Find the area of the region between $y = e^x$ and $y = 1/(1+x)$ on the interval $[0, 1]$.



1. Check for intersection points (verify algebraically that $x = 0$ is the only intersection by setting $e^x = \frac{1}{x+1}$).
2. Decide which function is on top ($f(x)$) and which function is on bottom ($g(x)$).
3. Calculate $\int_0^1 f(x) - g(x) dx$.

Check: What if you get a negative answer?

Example 2:
Find the area of the region bounded by $y = x^2 - 2x$ and $y = 4 - x^2$.

1. Check for intersection points (Solve $x^2 - 2x = 4 - x^2$). There will be two, a and b; this is where the functions cross.
2. Between this two points, which function is on top ($f(x)$) and which function is on bottom ($g(x)$).
3. Calculate $\int_a^b f(x) - g(x)dx$.

Check: What if you get a negative answer?

Example 3
Find the area between $\sin x$ and $\cos x$ on $[-3\pi/4, 5\pi/4]$.

(Hint: There are several places where $\sin(x) = \cos(x)$. For example, $x = \pi/4$.)

Example 4
Calculate the area under the curve $y = \arccos(x)$ from $x = 0$ to $x = 1$.

Hint: Since we don't know $\int \arccos(x)\ dx$, use the fact that $y = \arccos(x)$ if and only if $\cos(y) = x$.
(1) Draw graphs of both $y = \arccos(x)$ and $x = \cos(y)$ on separate axes (the first with x on the horizontal axis, and the second with y on the horizontal axis).
(2) What integral, involving $\cos(y)$ (and endpoints for y's instead of x's, and with a dy instead of a dx) will compute the same area as $\int_0^1 \arccos(x)\ dx$?

Example 5
Calculate the area under the curve $y = \arcsin(x)$ from $x = 0$ to $x = 1$.

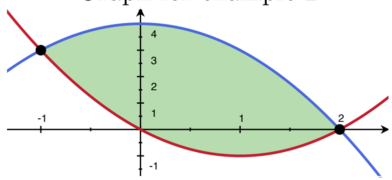
Hint: Similar to Example 4, but be careful! Be sure to draw the pictures before writing down the corresponding integrals!

Answers

Example 1: $e - 1 - \ln(2)$

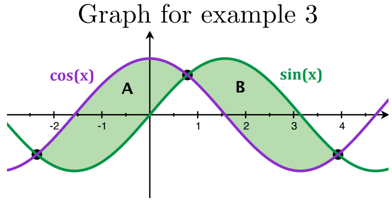
Example 2: 9

Graph for example 2



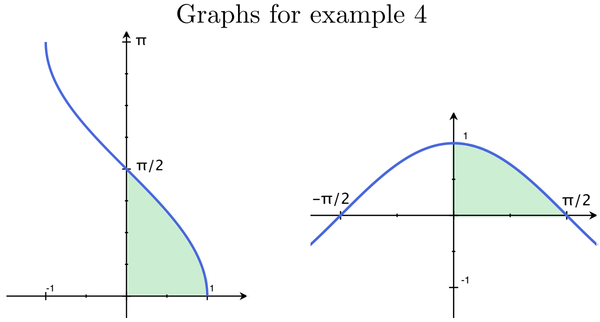
Example 3: $4\sqrt{2}$

Graph for example 3



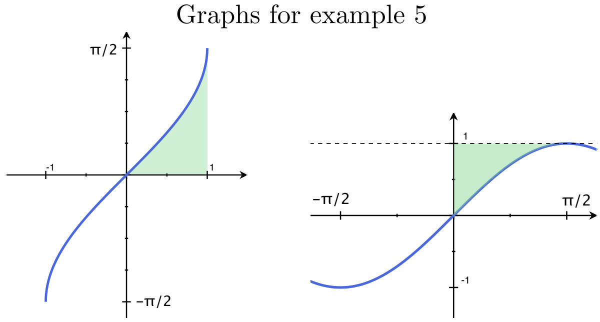
Example 4: 1

Graphs for example 4



Example 5: $\frac{\pi}{2} - 1$

Graphs for example 5



Extra practice: Areas using definite integrals

1. Find the area of the region bounded by the curve $xy - 3x - 2y - 10 = 0$, the x-axis, and the lines $x = 3$ and $x = 4$.

2. Find the area lying below the x-axis and above the parabola $y = 4x + x^2$.

3. Graph the curve $y = 2\sqrt{9 - x^2}$ and determine the area enclosed between the curve and the x-axis.

4. Find the area bounded by the curve $y = x(x - 3)(x - 5)$, the x-axis and the lines $x = 0$ and $x = 5$.

5. Find the area enclosed between the curve $y = \sin 2x$, $0 \le x \le \pi/4$ and the axes.

6. Find the area enclosed between the curve $y = \cos 2x$, $0 \le x \le \pi/4$ and the axes.

7. Find the area enclosed between the curve $y = 3\cos x$, $0 \le x \le \pi/2$ and the axes.

8. Show that the ratio of the areas under the curves $y = \sin x$ and $y = \sin 2x$ between the lines $x = 0$ and $x = \pi/3$ is $2/3$.

9. Find the area enclosed between the curve $y = \cos 3x$, $0 \le x \le \pi/6$ and the axes.

10. Find the area enclosed between the curve $y = \tan^2 x$, $0 \le x \le \pi/4$ and the axes.

11. Find the area enclosed between the curve $y = \csc^2 x$, $0 \le x \le \pi/4$ and the axes.

12. Compare the areas under the curves $y = \cos^2 x$ and $y = \sin^2 x$ between $x = 0$ and $x = \pi$.

13. Graph the curve $y = x/\pi + 2\sin^2 x$ and find the area between the x-axis, the curve and the lines $x = 0$ and $x = \pi$.

14. Find the area bounded by $y = \sin x$ and the x-axis between $x = 0$ and $x = 2\pi$.

15. Find the area of the region bounded by the parabola $y^2 = 4x$ and the line $y = 2x$.

16. Find the area bounded by the curve $y = x(2 - x)$ and the line $x = 2y$.

17. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

18. Calculate the area of the region bounded by the parabolas $y = x^2$ and $x = y^2$.

19. Find the area of the region included between the parabola $y^2 = x$ and the line $x + y = 2$.

20. Find the area of the region bounded by the curves $y = \sqrt{x}$ and $y = x$.

21. Find the area of the part of the first quadrant which is between the parabola $y^2 = 3x$ and the circle $x^2 + y^2 - 6x = 0$.

22. Find the area of the region between the curves $y^2 = 4x$ and $x = 3$.

23. Use integration to find the area of the triangular region bounded by the lines $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.

24. Find the area bounded by the parabola $x^2 - 2 = y$ and the line $x + y = 0$.

25. Find the area bounded by the curves $y = 3x - x^2$ and $y = x^2 - x$.

26. Graph the curve $y = (1/2)x^2 + 1$ and the straight line $y = x + 1$ and find the area between the curve and the line.

27. Find the area of the region between the parabolas $4y^2 = 9x$ and $3x^2 = 16y$.

28. Find the area of the region between the curves $x^2 + y^2 = 2$ and $x = y^2$.

29. Find the area of the region between the curves $y = x^2$ and $x^2 + 4(y - 1) = 0$.

30. Find the area of the region between the circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$.

31. Find the area of the region enclosed by the parabola $y^2 = 4ax$ and the line $y = mx$.

32. Find the area between the parabolas $y = 4ax$ and $y^2 = 4ay$.

33. Find the area of the region between the two circles $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 = 1$.

34. Find the area bounded by the curves $y = x$ and $y = x^3$.

35. Graph $y = \sin x$ and $y = \cos x$ for $0 \le x \le \pi/2$ and find the area enclosed by them and the x-axis.