Antiderivatives and Initial Value Problems

Warm up

If $\frac{d}{dx}f(x) = 2x$, what is $f(x)$?

Can you think of another function that $f(x)$ could be?

If $\frac{d}{dx}f(x) = 3x^2 + 1$, what is $f(x)$?

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Definition

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2. Another antiderivative of $f(x) = 2x$ is $F(x) = x^2 + 1$.
3. There are *lots* of antiderivatives of $f(x) = 2x$ which look like $F(x) = x^2 + C$.

Suppose that h is differentiable in an interval I,
and $h'(x) = 0$ for all x in I.

Then h is a constant function!
i.e. $h(x) = C$ for all $x \in I$, where C is a constant.

So, if $F(x)$ is one antiderivative of $f(x)$, then any other antiderivative must be of the form $F(x) + C$.

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Example: **All** of the antiderivatives of $f(x) = 2x$ look like

$$F(x) = x^2 + C$$

for some constant C.

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Example:

$$\int 2x \; dx = x^2 + C.$$

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$$\int x^2 \; dx = \frac{1}{3}x^3 + C, \qquad \text{because} \qquad \frac{d}{dx}(\frac{1}{3}x^3 + C) = \frac{1}{3} * 3x^2 = x^2$$

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$$\int x^5 \; dx =$$

$$\int x^{-3} \; dx =$$

$$\int x^k \; dx =$$

Some important basic integrals

$$\int x^k \, dx = \frac{1}{k+1}x^{k+1} + C \qquad \text{if } k \neq -1$$

$$\int x^{-1} \, dx =$$

$$\int \sin(x) \, dx =$$

$$\int \cos(x) \, dx =$$

$$\int e^x \, dx =$$

$$\int \sec^2(x) \, dx =$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx =$$

Theorem (Opposite of sum and constant rules)

Suppose the functions f and g both have antiderivatives on the interval I. Then for any constants a and b, the function $af + bg$ has an antiderivative on I and

$$\int \big(a * f(x) + b * g(x)\big) dx = a \int f(x) dx + b \int g(x) dx$$

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For example, $y = e^{5x}$ is a solution to the differential equation above since

$$\frac{d}{dx} e^{5x} = 5e^{5x},$$

so

$$\frac{dy}{dx} - 5y = (5e^{5x}) - 5(e^{5x}) = 0 \quad \checkmark$$

Simplest differential equations: antiderivatives

Finding an antiderivative can also be thought of as solving a differential equation:

"Solve the differential equation $\frac{d}{dx}y = x^2$."

$$\text{Answer:} \quad y = \int x^2 dx = \frac{1}{3}x^3 + C.$$

$$\text{Check:} \quad \frac{d}{dx}\frac{1}{3}x^3 + C. = \frac{1}{3} * 3 * x^2 + 0 = x^2 \quad \checkmark$$

Examples

(1) Solve the differential equation $y' = 2x + \sin(x)$.

(2) Check that $\cos(x) + \sin(x)$ is a solution to $\dfrac{d^2y}{dx^2} + y = 0$.

Initial value problems

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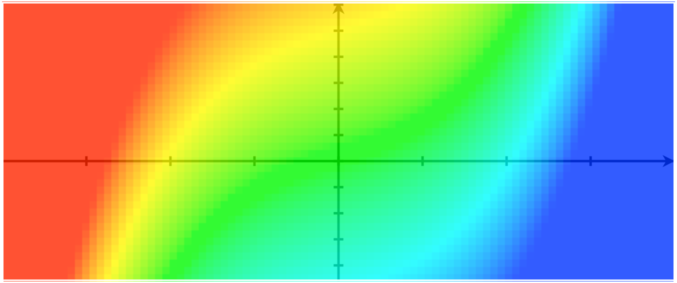

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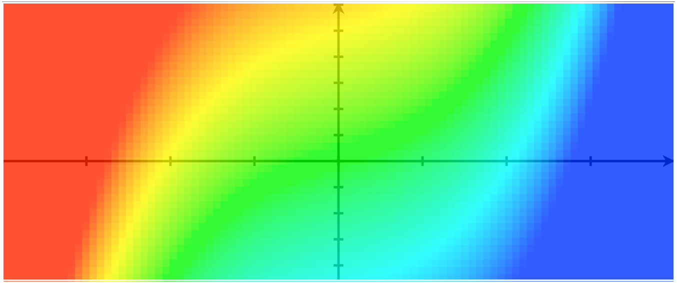

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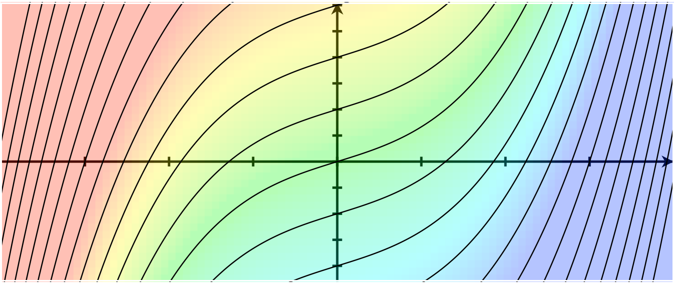

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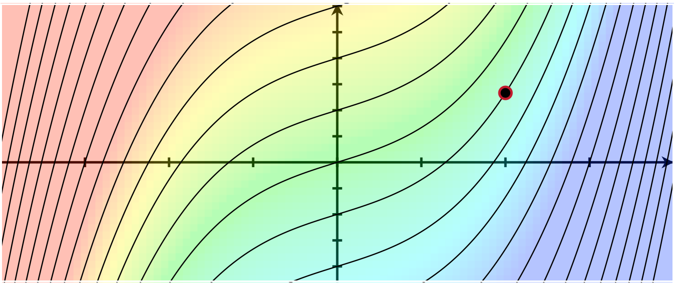

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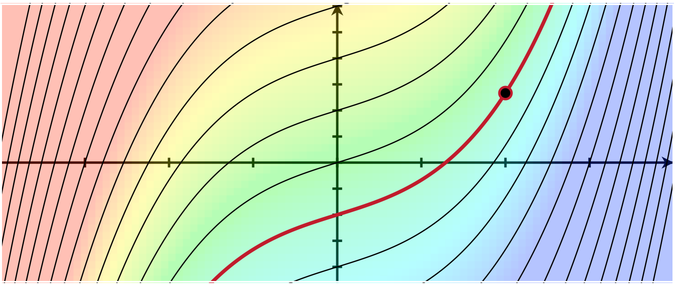

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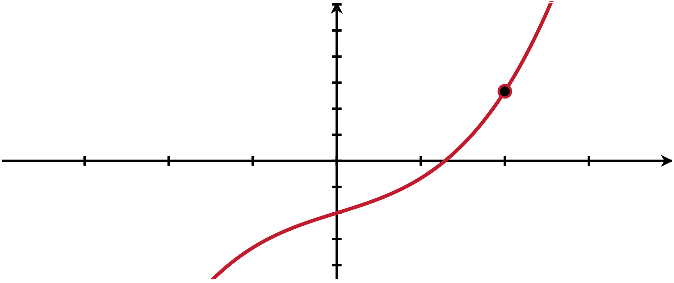
particular solution:	$y = \frac{1}{3}x^3 + x - 2$



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Definition

An **initial-value problem** is a differential equation together with enough additional conditions to specify the constants of integration that appear in the general solution.

The **particular solution of the problem** is then a function that satisfies both the differential equation and also the additional conditions.

Solve the initial value problem

$$\frac{dy}{dx} = 2x + \sin(x)$$

subject to $y(0) = 0$.

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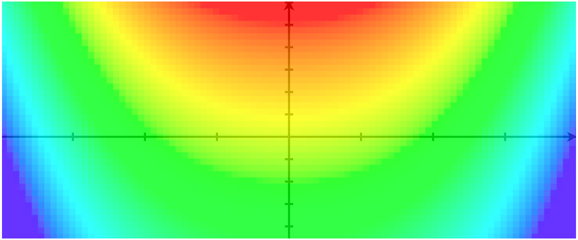

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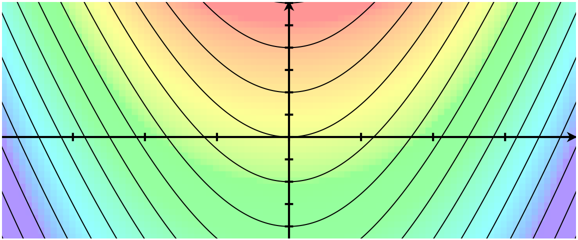

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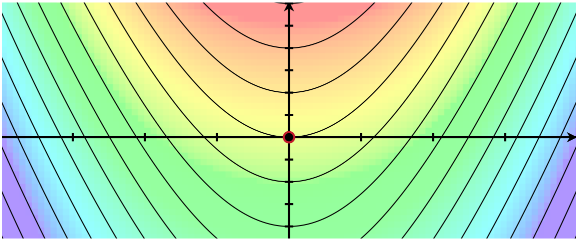

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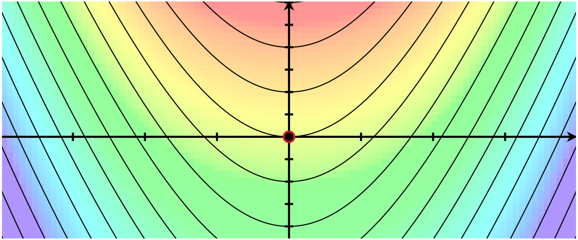

Algebraically: get a particular solution by solving

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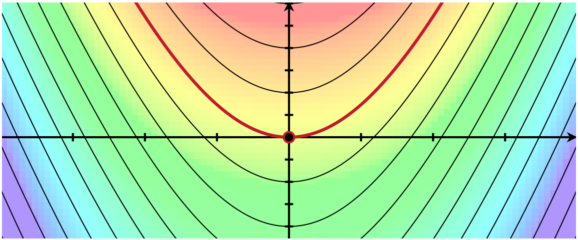

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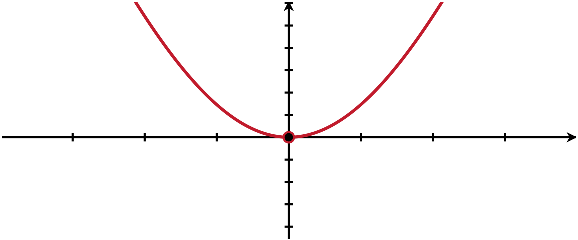

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Step 1: Calculate the antiderivative of $\cos(x)$ to find the general solution for y'.

Step 2: Plug in the values $y'(\frac{\pi}{2}) = 2$ to calculate C.

Step 3: Write down the *particular* solution for y'.

Step 4: Calculate the antiderivative of your particular solution in Step 3 to find the *general solution for y*.

Step 5: Plug in the values $y(\frac{\pi}{2}) = 3\pi$ to solve for the new constant.

Step 6: Write down the *particular* solution for y.

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An object dropped from a cliff has acceleration $a = -9.8\ m/sec^2$ under the influence of gravity. What is the function $s(t)$ that models its height at time t?

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