Optimization

Warm up

Sketch the graph of

$$f(x) = (x-3)(x-2)(x-1) = x^3 - 6x^2 + 11x - 6$$

over the interval $[1, 4]$. Mark any critical points and inflection points. What is the absolute maximum over this interval? What is the absolute minimum over this interval?
[useful value: $\sqrt{3}/3 \approx .6$]

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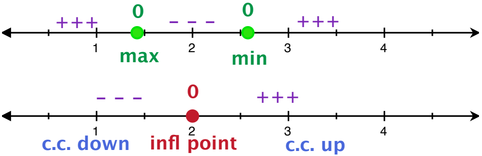
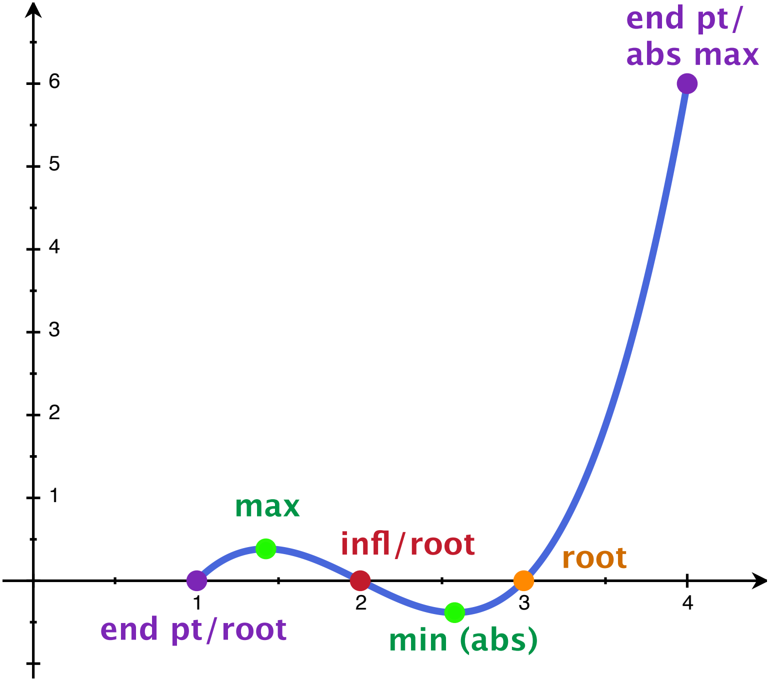
$$f'(x) = 3x^2 - 12x + 11 = 3\left(x - (2 - \sqrt{3}/3)\right)\left(x - (2 + \sqrt{3}/3)\right)$$

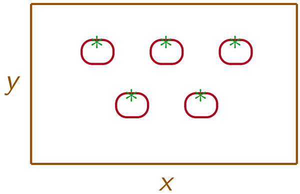
$$f''(x) = 6x - 12 = 6(x-2)$$



$f(2 - \sqrt{3}/3) \approx 0.385$
$f(2 + \sqrt{3}/3) \approx -0.385$
$f(1) = 0$
$f(4) = 6$


end pt/
abs max
6
5
4
3
2
1
max
infl/root
root
1
2
3
4
end pt/root
min (abs)

Suppose you want to fence off a garden, and you have 100m of fence. What is the largest area that you can fence off?



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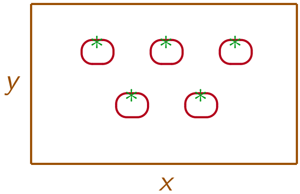

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Know: $2x + 2y = 100$ Want: Maximize $A = xy$

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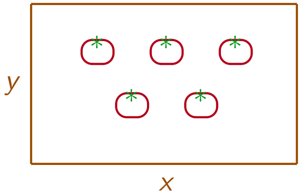

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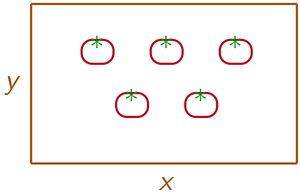

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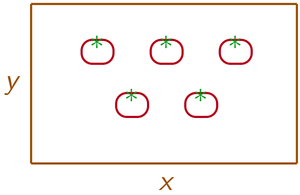

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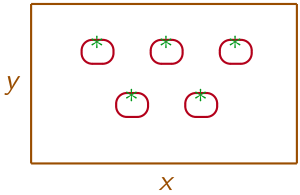

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Domain: $0 \le x \le 50$

New problem: Maximize $A(x) = 50x - x^2$ over the interval $0 < x < 50$.

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Solution...

Three strategies:

(1) First derivative test:

(2) Pretend we're on a closed interval, then throw out the endpoints:

(3) Second derivative test:

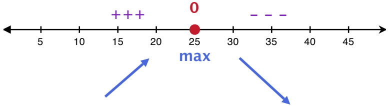
New problem: Maximize $A(x) = 50x - x^2$ over the interval $0 < x < 50$.

Solution: $A'(x) = 50 - 2x$

So the only critical point is when $50 - 2x = 0$, so $\boxed{x = 25}$.

Three strategies:

(1) First derivative test:



So $A(25) = 625$ is maximal.

(2) Pretend we're on a closed interval, then throw out the endpoints:

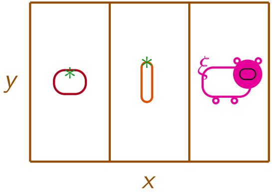
x	$A(x)$	
25	625	max!
0	0	min
50	0	min

Since the maximum is not at one of the points I have to throw out, it must me a maximum on the open interval (there is no absolute minimum over the open interval $(0, 50)$).

(3) Second derivative test:

$$A''(x) = -2 < 0 \text{ so } A(25) = 625 \text{ must be a maximum.}$$

Now suppose, instead, you want to divide your plot up into three equal parts:



If you still only have 100 m of fence, what is the largest area that you can fence off?

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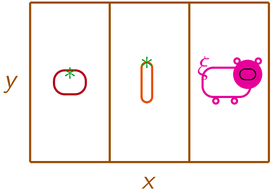


If you still only have 100 m of fence, what is the largest area that you can fence off?

Solution:

Constraint: $2x + 4y = 100$, so $y = 25 - \frac{1}{2}x$.

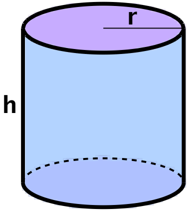
Maximize: $A = xy$ over $0 < x < 50$.

Plug in constraint: $A(x) = x(25 - x/2) = 25x - x^2/2$

Find critical points: $0 = A'(x) = 25 - x$, so $x = 25$.

Second derivative test: $A''(x) = -1 < 0$ so $\boxed{A(25) = 25 * 12.5}$ is a maximum.

Suppose you want to make a can which holds about 16 ounces (28.875 in^3). If the material for the top and bottom of the can costs 4 $¢/in^2$ and the material for the sides of the can costs 3 $¢/in^2$. What is the minimum cost for the can?



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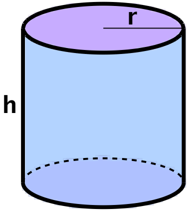

Put into math:

Constraint: $V = \pi r^2 h = 28.875$.

Cost: $4 * (\text{SA of top} + \text{bottom}) + 3 * (\text{SA of side})$

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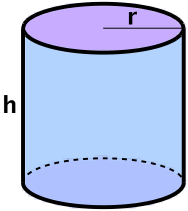

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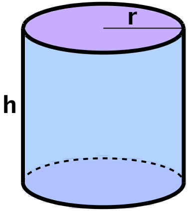

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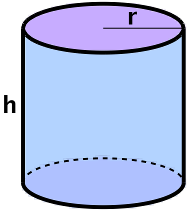

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Get into one variable: Use the constraint!

$$\pi r^2 h = 28.875 \implies h = \frac{28.875}{\pi} r^{-2}$$

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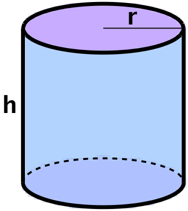

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$$\pi r^2 h = 28.875 \implies h = \frac{28.875}{\pi} r^{-2} \implies C(r) = 8\pi r^2 + 6\pi r \left(\frac{28.875}{\pi} r^{-2}\right)$$

So

$$\boxed{C(r) = 8\pi r^2 + 6 * 28.875 r^{-1}}$$

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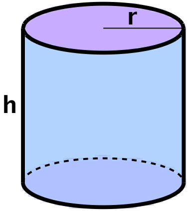

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(Domain: $r > 0$)

New problem: Minimize $C(r) = 8\pi r^2 + 6 * 28.875r^{-1}$ for $r > 0$.

[hint: If you don't have a calculator, use the second derivative test!]

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Solution: $(6 * 28.875 = 173.25)$

$$C'(r) = 16\pi r - 173.25 r^{-2} = \frac{1}{r^2}\left(16\pi r^3 - 173.25\right)$$

Critical point: $C'(r) = \sqrt[3]{\frac{173.25}{16\pi}} \approx 1.031$

Second derivative test: $C''(r) = 16\pi + 173.25 r^{-3} > 0$ when $r > 0$,

so $C(r)$ is concave up, and so $C\left(\sqrt[3]{\frac{173.25}{16\pi}}\right)$ is a minimum.

Minimal value: $C\left(\sqrt[3]{\frac{173.25}{16\pi}}\right) \approx 80.2041$¢