Example
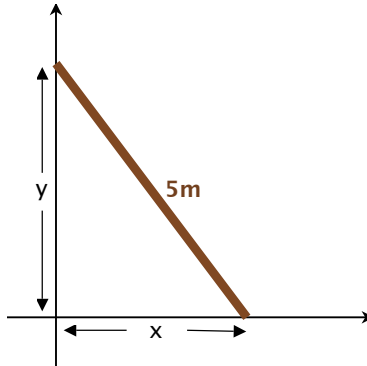
Suppose you has a 5m ladder resting against a wall.

Move the base out at 2 m/s: $\frac{dx}{dt} = 2$

How fast does the top move down the wall? $\frac{dy}{dt} = ??$

To solve, we need to relate the variables: $x^2 + y^2 = 5^2$

$$0 \leq x \leq 5$$

Problem:
If $x^2 + y^2 = 5^2$ for $0 \leq x \leq 5$, and $\frac{dx}{dt} = 2$, what is $\frac{dy}{dt}$?

Differentiate:

$$0 = \frac{d}{dt} 5^2 = \frac{d}{dt} \left(x^2 + y^2 \right)$$
$$= 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$
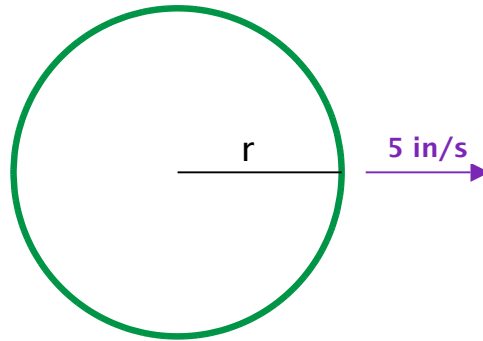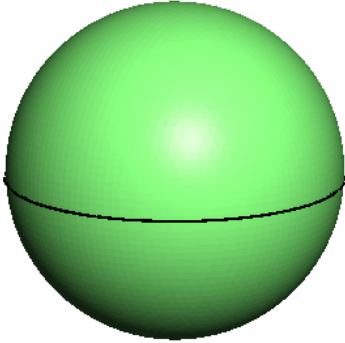$$= 2x * 2 + 2 \left(\sqrt{25 - x^2} \right) \frac{dy}{dt}$$

So

$$\boxed{\frac{dy}{dt} = \frac{-2x}{\sqrt{25 - x^2}}}$$

Notice:
(1) $\frac{dy}{dt} < 0$ (y is decreasing)   and   (2) $\lim_{x \to 5^-} \frac{dy}{dt} \to -\infty$

Example

Suppose you have a sphere whose radius is growing at a rate of 5in/s. How fast is the volume growing when the radius is 3in?
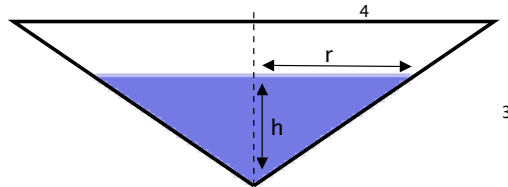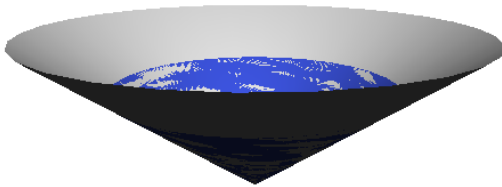
Relating equation: $V = \frac{4}{3}\pi r^3$

Take a derivative: $\frac{dV}{dt} = \frac{4}{3}\pi * 3r^2 * \frac{dr}{dt}$

Substitute in the known values:
$\frac{dV}{dt} = 4\pi * 3^2 * 5 = \boxed{4 * 9 * 5\pi \text{ in}^3/s}$

Take an upside-down cone-shaped bowl, with a radius of 4in at the top and a total height of 3in fill it with water at a rate of $1/2$ in^3/min. How fast is the height of water increasing when h=2in?

Volume of a cone: $V = \frac{\pi}{3}R^2H$
Volume of a water: $V = \frac{\pi}{3}r^2h$
Relate r and h: $r/h = 4/3$ so $r = \frac{4}{3}h$
Finally, equation to differentiate: $V = \frac{\pi}{3}\left(\frac{4}{3}h\right)^2 h = \frac{\pi 16}{27}h^3$

$$\frac{1}{2} = \frac{dV}{dt} = \frac{\pi 16}{27} * 3h^2 \frac{dh}{dt} = \frac{\pi 16}{9}(2)^2 \frac{dh}{dt}$$

So $\boxed{\frac{dh}{dt}\big|_{h=2} = \frac{9}{128\pi}}$

Strategy:

1. Find an equation which relates the functions you need.

   (a) Sometimes you'll need to draw pictures.

   (b) Sometimes you'll have to reduce the number of variables/functions to get it down to
      (i) the function from the rate you know,
      (ii) the function from the rate you want, and
      (iii) maybe the variable from the rate you know and want (t in the last 3 examples).

2. Take a derivative using implicit differentiation.

3. Plug in the values you know.

4. Solve for the rate you want.

One more example: (from extra problems)
10.   A boat is pulled into a dock by a rope attached to the bow of the boat a passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s how fast is the boat approaching the dock when it is 8 m from the dock?

On your own: (from extra problems)
5. A balloon which always remains spherical is being inflated by pumping in 900 cubic centimeters of gas per second. Find the rate at which the radius of the balloon is increasing when the radius is 15cm.
11. Gravel is being dumped from a conveyor belt at a rate of 30 ft^3/min and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?
9. A lighthouse is on a small island 3 km away from the nearest point P on a straight shoreline and its light turns four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P? [hint: 4rpm means that some angle is changing at $4 * 2\pi$ radians per minute]

Extra practice: Related rates

1. Relate the change of the volume of a sphere of radius r versus time t to the change in the radius with respect to time.

2. Find the rate of change of the volume of a cylinder of radius r and height h with respect to a change in the radius, assuming the height is also a function of r.

3. Find the rate of change of the curved surface of a cone of radius r and height h with respect to a change in the radius, assuming the height is also a function of r.

4. The side of a square is increasing at the rate of 0.2 cm/s. Find the rate of increase of the perimeter of the square.

5. A balloon which always remains spherical is being inflated by pumping in 900 cubic centimeters of gas per second. Find the rate at which the radius of the balloon is increasing when the radius is 15cm.

6. The surface area of a spherical bubble is increasing at 2 cm^2/s. When the radius of the bubble is 6 cm at what rate is the volume of the bubble increasing?

7. The bottom of a rectangular swimming pool is 25×40 meters. Water is pumped into the tank at the rate of 500 cubic meters per minute. Find the rate at which the level of the water in the tank is rising.

8. A streetlight is at the top of a 15 foot tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path.

    (a) How fast is the tip of his shadow moving when he is 40 feet from the pole?

    (b) How fast is his shadow lengthening at that point?

9. A lighthouse is on a small island 3 km away from the nearest point P on a straight shoreline and its light turns four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P?

10. A boat is pulled into a dock by a rope attached to the bow of the boat a passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s how fast is the boat approaching the dock when it is 8 m from the dock?

11. Gravel is being dumped from a conveyor belt at a rate of 30 ft^3/min and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?

12. Water is dripping from a tiny hole in the vertex in the bottom of a conical funnel at a uniform rate of 4 cm^3/s. When the slant height of the water is 3 cm, find the rate of decrease of the slant height of the water, given that the vertical angle of the funnel is 120°.

13. Water is leaking out of an inverted conical tank at a rate of $10,000$ cm^3/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water height is rising at a rate of 20 cm/min when the hieght of the water is 2 m, find the rate at which water is being pumped into the tank.

14. Oil is leaking from a cylindrical drum at a rate of 16 milliliters per second. If the radius of the drum is 7 cm and its height is 60 cm find the rate at which the level of oil is changing when the oil level is 18 cm.

15. A ladder 10 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a speed of 2 ft/s, how fast is the angle between the top of the ladder and the wall changing when the angle is $\pi/4$ radians?

16. A ladder 13 meters long is leaning against a wall. The bottom of the ladder is pulled along the ground away from from the wall at the rate of 2 m/s. How fast is its height on the wall decreasing when the foot of the ladder is 5 m away from the wall?

17. A man is moving away from a 40 meter tower at a speed of 2 m/s. Find the rate at which the angle of elevation of the top of the tower is changing when he is at a distance of 30 meters from the foot of the tower. Assume that the eye level of the man is 1.6 meters from the ground.

18. Find the angle which increases twice as fast as its sine.

19. a rocket launching pad. A rocket rises vertically and its speed is 600 ft/s when it has risen 3000 feet.

   (a) How fast is the distance from the television camera to the rocket changing at that moment?
   (b) How fast is the camera's angle of elevation changing at that same moment?