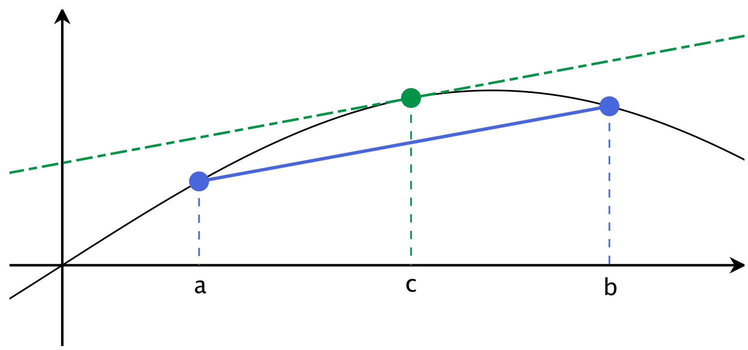
The Mean Value Theorem

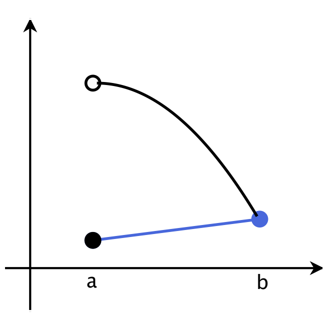
Theorem

Suppose that f is defined and continuous on a closed interval $[a, b]$, and suppose that f' exists on the open interval (a, b). Then there exists a point c in (a, b) such that

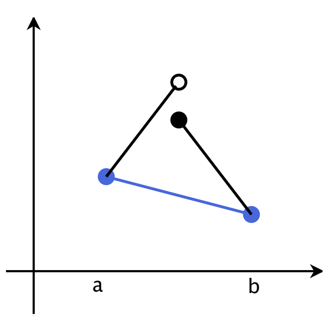
$$\frac{f(b) - f(a)}{b - a} = f'(c).$$



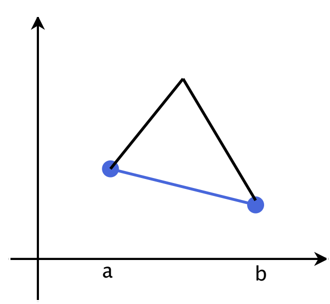
Bad examples



Discontinuity
at an endpoint



Discontinuity
at an interior point



No derivative
at an interior point

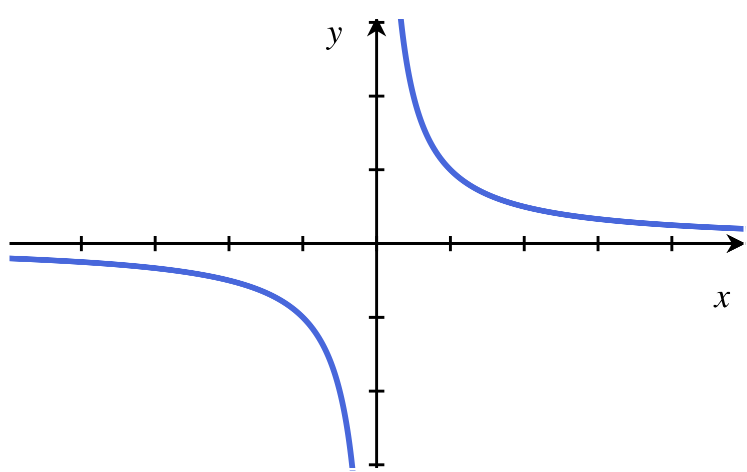
Examples

Does the mean value theorem apply to $f(x) = |x|$ on $[-1, 1]$?

How about to $f(x) = |x|$ on $[1, 5]$?

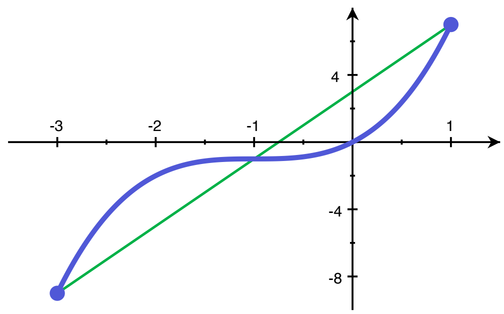
Example

Under what circumstances does the Mean Value Theorem apply to the function $f(x) = 1/x$?



Example

Verify the conclusion of the Mean Value Theorem for the function $f(x) = (x+1)^3 - 1$ on the interval $[-3, 1]$.

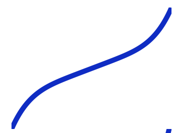
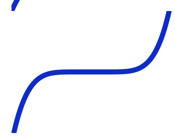
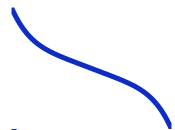
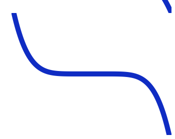


Step 1: Check that the conditions of the MVT are met.

Step 2: Calculate the slope m of the line joining the two endpoints.

Step 3: Solve the equation $f'(x) = m$.

Intervals on increase/decrease

Formally,		$\frac{f(x+h)-f(x)}{h}$	$\lim_{h\to 0} \sim$
f is *increasing* if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.		pos.	pos. or 0 (non-neg)
f is *nondecreasing* if $f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$.		non-neg.	non-neg.
f is *decreasing* if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.		neg.	non-pos.
f is *nonincreasing* if $f(x_1) \geq f(x_2)$ whenever $x1 < x2$.		non-pos.	non-pos.

So we can calculate some of the "shape" of $f(x)$ by knowing when its derivative is positive, negative, and 0!

Sign of the derivative

If $f(x)$ is **increasing**, what is the sign of the derivative?
Look at the difference quotient:

$$\frac{f(x+h)-f(x)}{h}$$

The derivative is a two-sided limit, so we have two cases:

Case 1: h is positive.
So $x+h > x$, which implies $f(x+h)-f(x) > 0$.
So
$$\frac{f(x+h)-f(x)}{h} > 0.$$

Case 2: h is negative.
So $x+h < x$, which implies $f(x+h)-f(x) < 0$.
So
$$\frac{f(x+h)-f(x)}{h} > 0.$$

So the difference quotient is positive!

Example

On what interval(s) is the function $f(x) = x^3 + x + 1$ increasing or decreasing?

Step 1: Calculate the derivative.

Step 2: Decide when the derivative is positive, negative, or zero.

Step 3: Bring that information back to $f(x)$.

Example

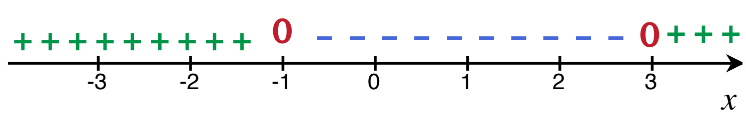
Find the intervals on which the function $f(x) = 2x^3 - 6x^2 - 18x + 1$ is increasing and those on which it is decreasing.

Step 1: Calculate the derivative.

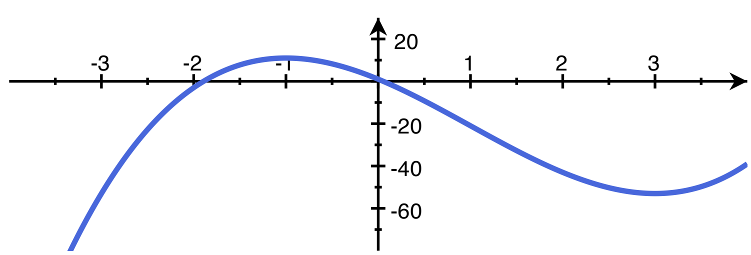
Step 2: Decide when the derivative is positive, negative, or zero.

Step 3: Bring that information back to $f(x)$.

$f'(x)$:



$f(x)$:

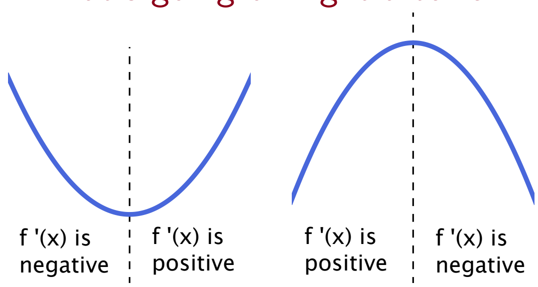


If f is continuous on a closed interval $[a, b]$, then there is a point in the interval where f is largest (**maximized**) and a point where f is smallest (**minimized**).

The maxima or minima will happen either

1. at an endpoint, or
2. at a **critical point**, a point c where $f'(c) = 0$

what's going on right before c?
what's going on right after c?

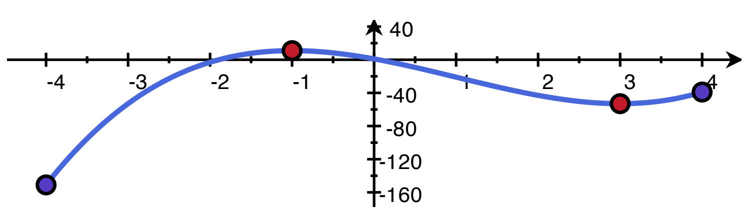


Example

For the function $f(x) = 2x^3 - 6x^2 - 18x + 1$, let us find the points in the interval $[-4, 4]$ where the function assumes its maximum and minimum values.

$$f'(x) = 6x^2 - 12x - 18 = 6(x-3)(x+1)$$

x	$f(x)$
-1	11
3	53
-4	-151
4	-39

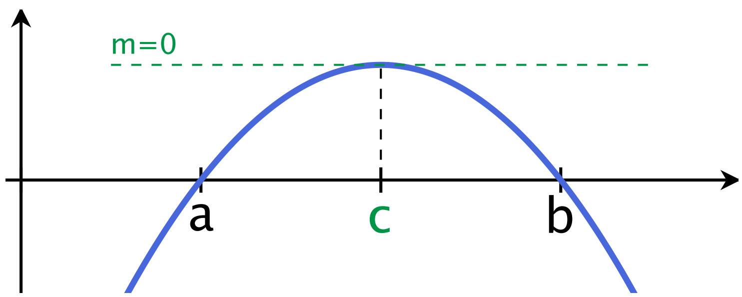


Rolle's Theorem

Theorem
Suppose that the function f is

- **continuous** *on the closed interval* $[a, b]$,
- **differentiable** *on the open interval* (a, b), *and*
- *a and b are both* **roots** *of f.*

Then there is at least one point c in (a, b) where $f'(c) = 0$.



(In other words, if g didn't jump, then it had to turn around)

Back to Newton's method

Remember: Newton's method helped us fine roots of functions.

> Pick an x_0 to start. To get x_{i+1}, follow the tangent line to $f(x)$ at x_i down to it's x-intercept. The x_i's get closer and closer to a root of f.

But how do we know when we've found all of them?
For example: Find the roots of $f(x) = x^5 - 3x + 1$.

If x_0 is...	-2	-1	0	1	2
then the x_i's get closer to...					
	-1.3888	-1.3888	0.3347	1.2146	1.2146

$x_0 =$	-.9	-.8	-.7	-.6	.5	.6	.7
$x_i \to$	-1.3...	1.2 ...	1.2...	0.3...	0.3...	0.3...	0.3...
$x_0 =$	-10	-20	-50	-100	-1000	-10000	
$x_i \to$	-1.3...	-1.3...	-1.3...	-1.3...	-1.3...	-1.3...	
$x_0 =$	10	20	50	100	1000	10000	100000
$x_i \to$	1.2...	1.2...	1.2...	1.2...	1.2...	1.2...	1.2...

After plugging in lots of x_0's, we've only found three roots. But there could be up to 5! How do we know we're not just very unlucky?

Use Rolle's Theorem to show that $f(x) = x^5 - 3x + 1$ has exactly three real roots!

Step 1: Show that there are at *most* three roots.

Step 2: Show that there are at *least* three roots.

Two methods:

(1) Use Newton's method to root out three roots, or

(2) find four points $f(x)$ which alternate signs, and use the intermediate value theorem.

(IVT: If $f(x)$ is cont. and $f(a) < C < f(b)$, then there's a c btwn. a and b where $f(c) = C$)

On your own:

1. **Do an analysis of increasing/decreasing on $f(x)$.**
   How many times does $f(x)$ turn around?
   Conclude: what is an upper bound on the number of roots?
2. **Find the heights of the critical points.**
   Using the intermediate value theorem, what is a lower bound on the number of roots? Can you do better if you also find the height of the function at a big positive number and a big negative number?
3. **Conclude: How many real roots does $f(x)$ have?**
4. **Bonus:**
   Using the approximations from before, sketch a graph.