# Newton's Method and Linear Approximations

Curves are tricky. Lines aren't.

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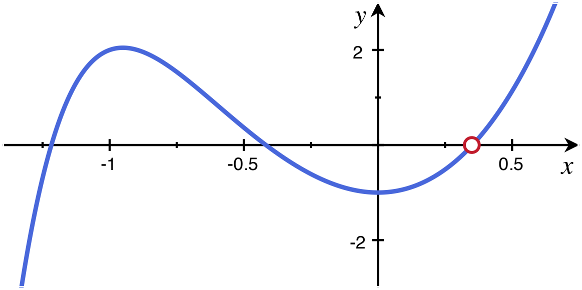
# Newton's Method for finding roots

**Goal: Where is $f(x) = 0$?**

$$f(x) = x^7 + 3x^3 + 7x^2 - 1$$



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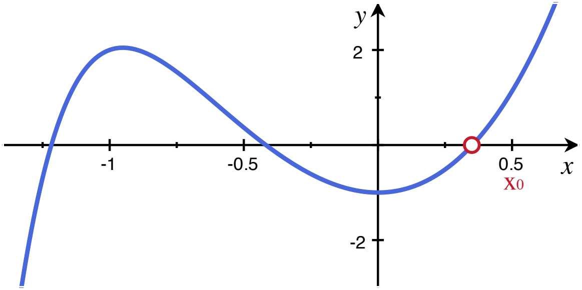

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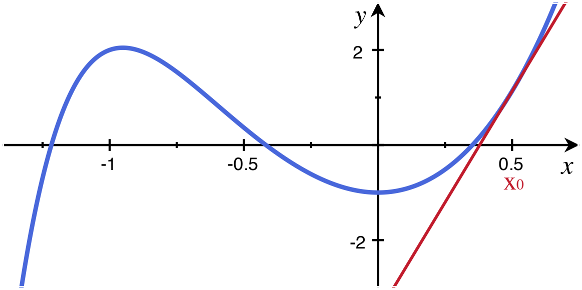

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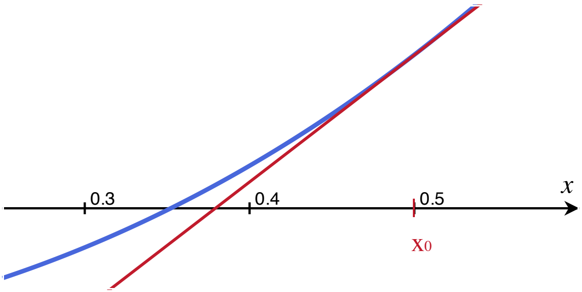

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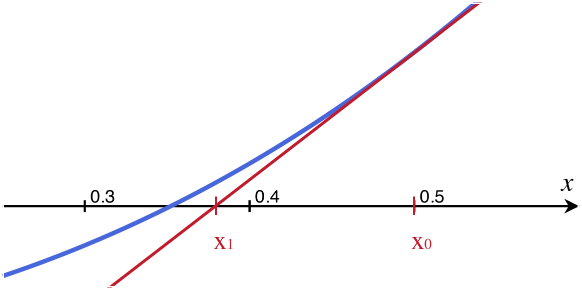

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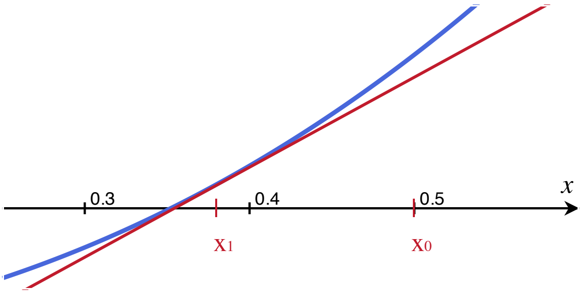

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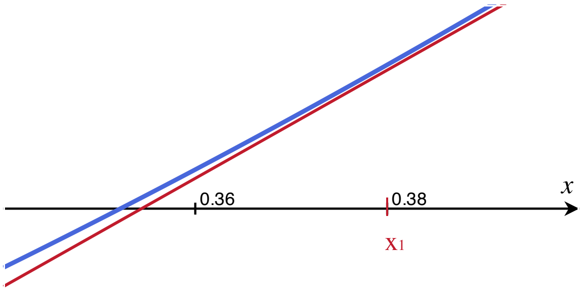

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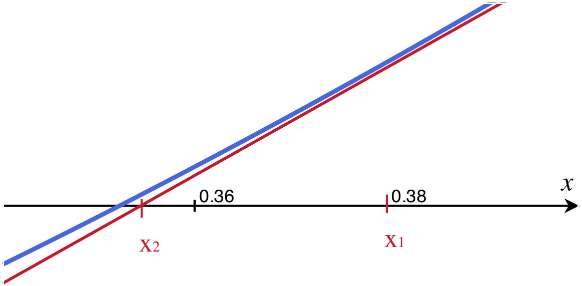

$$f(x) = x^7 + 3x^3 + 7x^2 - 1$$

$$f'(x) = 7x^6 + 9x^2 + 14x$$

| $i$ | $x_i$ | $f(x_i)$ | $f'(x_i)$ | tangent line | $x$-intercept |
|---|---|---|---|---|---|
| 0 | 0.5 | | | | |
| 1 | | | | | |
| 2 | | | | | |
| 3 | | | | | |

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| 0 | 0.5 | 1.133 | 9.359 | $y = 1.133 + 9.359(x - 0.5)$ | 0.379 |
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|---|---|---|---|---|---|
| 0 | 0.5 | 1.133 | 9.359 | $y = 1.133 + 9.359(x - 0.5)$ | 0.379 |
| 1 | 0.379 | 0.170 | 6.619 | $y = 0.170 + 6.619(x - 0.379)$ | 0.353 |
| 2 | | | | | |
| 3 | | | | | |

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| 1 | 0.379 | 0.170 | 6.619 | $y = 0.170 + 6.619(x - 0.379)$ | 0.353 |
| 2 | 0.353 | | | | |
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|---|---|---|---|---|---|
| 0 | 0.5 | 1.133 | 9.359 | $y = 1.133 + 9.359(x - 0.5)$ | 0.379 |
| 1 | 0.379 | 0.170 | 6.619 | $y = 0.170 + 6.619(x - 0.379)$ | 0.353 |
| 2 | 0.353 | 0.007 | 6.084 | $y = 0.007 + 6.084(x - 0.353)$ | 0.352 |
| 3 | | | | | |

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|---|---|---|---|---|---|
| 0 | 0.5 | 1.133 | 9.359 | $y = 1.133 + 9.359(x - 0.5)$ | 0.379 |
| 1 | 0.379 | 0.170 | 6.619 | $y = 0.170 + 6.619(x - 0.379)$ | 0.353 |
| 2 | 0.353 | 0.007 | 6.084 | $y = 0.007 + 6.084(x - 0.353)$ | 0.352 |
| 3 | 0.352 | | | | |

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| 1 | 0.379 | 0.170 | 6.619 | $y = 0.170 + 6.619(x - 0.379)$ | 0.353 |
| 2 | 0.353 | 0.007 | 6.084 | $y = 0.007 + 6.084(x - 0.353)$ | 0.352 |
| 3 | 0.352 | 0.00001 | 6.060 | $y = 0.00001 + 6.060(x - 0.352)$ | 0.352 |

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Step 1: Pick a place to start. Call it $x_0$.

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$$0 = f(x_0) + f'(x_0) * (x - x_0) \quad \text{to get} \quad x = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

This value is your $x_1$.

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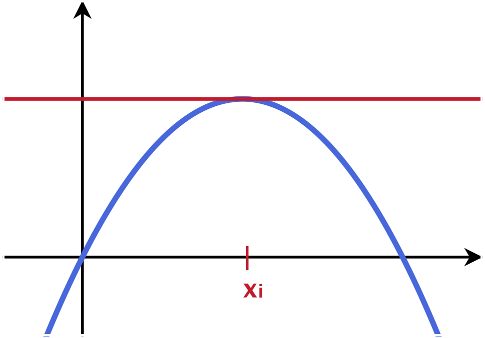
Step 4: Keep going until your $x_i$'s stabilize.
What they stabilize to is an approximation of your root!

# Caution!

Bad places to pick: Critical points! (where f'(x)=0)




Tangent line has no $x$-intercept!

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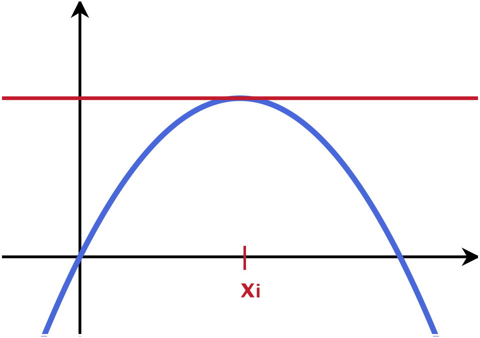
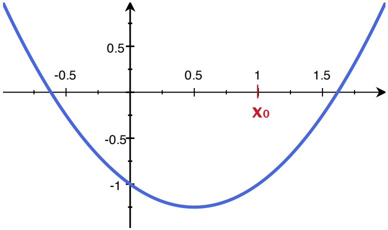



Tangent line has no $x$-intercept!

Even *near* critical points, the algorithm goes much slower.
Just stay away!

**You try:** Approximate a root of $f(x) = x^2 - x - 1$ near $x_0 = 1$.




$$f'(x) =$$

| $i$ | $x_i$ | $f(x_i)$ | $f'(x_i)$ | $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ |
|---|---|---|---|---|
| 0 | 1 | | | |
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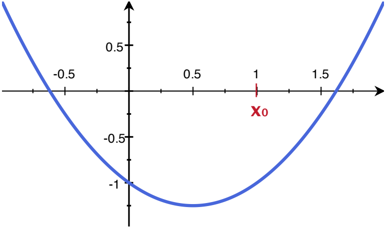


$$f'(x) = 2x - 1$$

| $i$ | $x_i$ | $f(x_i)$ | $f'(x_i)$ | $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ |
|---|---|---|---|---|
| 0 | 1 | $-1$ | 1 | 2 |
| 1 | 2 | 1 | 3 | $5/3 \approx 1.667$ |
| 2 | $5/3$ | $1/9$ | $7/3$ | $34/21 \approx 1.619$ |

# Back to the example:

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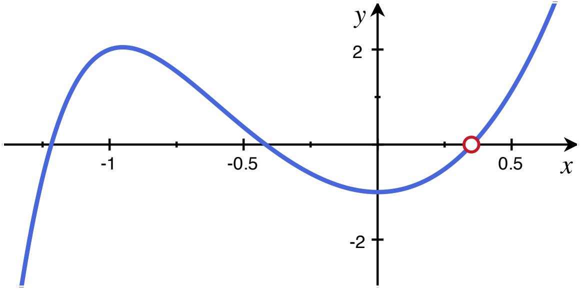

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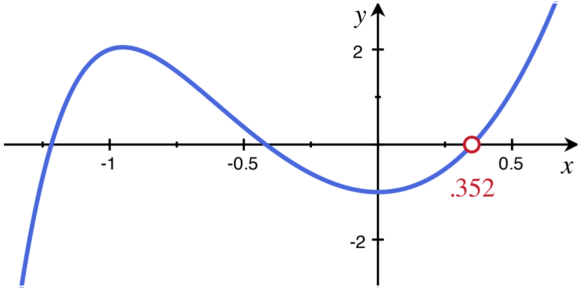


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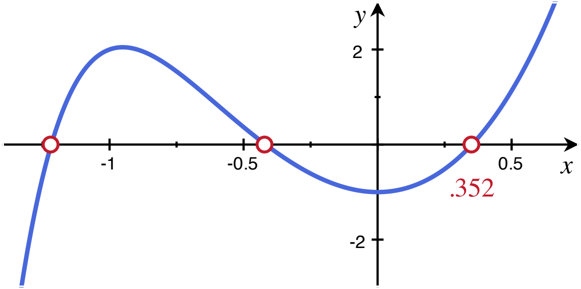


$r_1 \approx$ $r_2 \approx$ $r_3 \approx 0.352$

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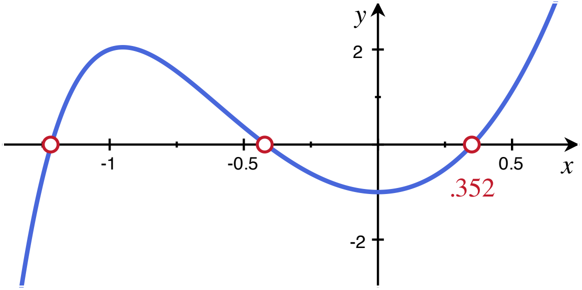


$r_1 \approx -1.217$ $\quad r_2 \approx -0.418$ $\quad r_3 \approx 0.352$

# Linear approximations of functions

**Goal: approximate functions**

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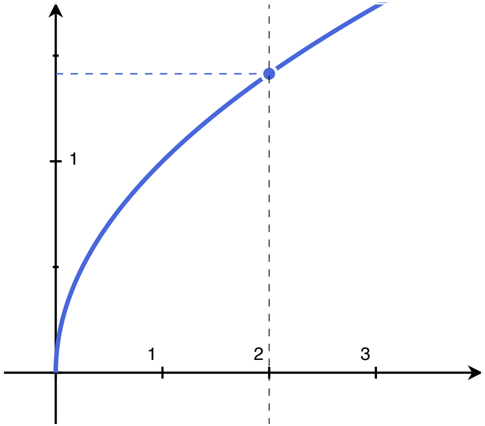
Example: approximate $\sqrt{2}$



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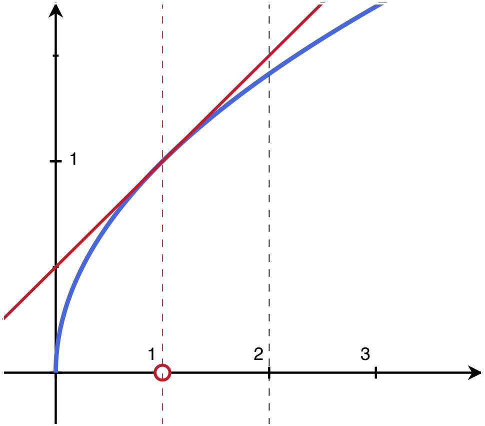


$$y = 1 + \frac{1}{2}(x-1)$$

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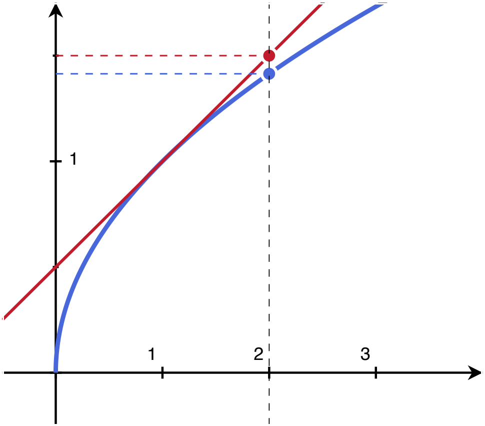


$$y = 1 + \frac{1}{2}(x - 1)$$
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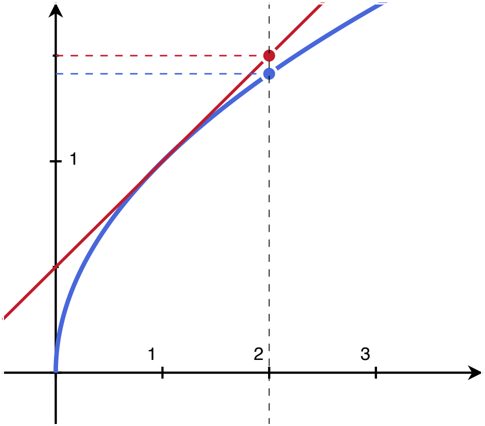


$$y = 1 + \frac{1}{2}(x - 1)$$

$$\sqrt{2} \approx 1 + \frac{1}{2}(2 - 1) = 1.5 \qquad (\sqrt{2} = 1.414...)$$

# Linear approximations

If $f(x)$ is differentiable at $a$, then the tangent line to $f(x)$ at $x = a$ is
$$y = f(a) + f'(a) * (x - a).$$

For values of $x$ *near* $a$, then
$$f(x) \approx f(a) + f'(a) * (x - a).$$

This is the *linear approximation* of $f$ about $x = a$. We usually call the line $L(x)$.

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Our last approximation told us

$$\sqrt{5} \approx L(5) = 1 + \frac{1}{2}(5 - 1) = 3$$

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Better: Use the linear approximation about $x = 4$!

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The tangent line is

$$L(x) = 2 + \frac{1}{4}(x - 4)$$

so

$$\sqrt{5} \approx L(5) = 2 + \frac{1}{4}(5 - 4) = \boxed{2.25}$$

Better! $(2.25^2 = 5.0625)$

# Even better approximations...

The linear approximation is **the** line which satisfies

$$L(a) = f(a) + f'(a)(a - a) = \boxed{f(a)}$$

**and**

$$L'(a) = \frac{d}{dx}\big(f(a) + f'(a)(x - a)\big) = \boxed{f'(a)}$$

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A **better** approximation might be a quadratic polynomial $p_2(x)$ which **also** satisfies $p_2''(a) = f''(a)$:

$$\boxed{p_2(x) = f(a) + f'(a)(x - a) + \tfrac{1}{2}f''(a)(x - a)^2}$$

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or a cubic polynomial $p_3(x)$ which also satisfies $p_3^{(3)}(a) = f^{(3)}(a)$:

$$\boxed{p_3(x) = f(a) + f'(a)(x - a) + \tfrac{1}{2} f''(a)(x - a)^2 + \tfrac{1}{2*3} f^{(3)}(a)(x - a)^3}$$

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and so on...

These approximations are called **Taylor polynomials** (read §2.14)