More on implicit differentiation

We can now take derivatives of things that look like

$$x^2 + y^2 = 1 \qquad \text{or } e^y = xy$$

Ex 1: If $x^2 + y^2 = 1$, then take $\frac{d}{dx}$ of both sides to find

$$2x + 2y * \frac{dy}{dx} = 0$$

so

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

Ex 2: If $e^y = x$, then take $\frac{d}{dx}$ of both sides to find

$$\frac{dy}{dx} * e^y = x\frac{dy}{dx} + y.$$

So

$$y = \frac{dy}{dx} * e^y - x\frac{dy}{dx} = \frac{dy}{dx}(e^y - x)$$

So $\boxed{\frac{dy}{dx} = \frac{y}{e^y - x}}$

Every time:
(1) Take $\frac{d}{dx}$ of both sides.
(2) Add and subtract to get the $\frac{dy}{dx}$ on one side and everything else on the other.
(3) Factor out $\frac{dy}{dx}$ and divide both sides by its coefficient.

More on implicit differentiation

We can also take derivatives versus other variables:
Example Suppose $\cos(y) = x + y$.

1. Calculate $\frac{dy}{dx}$
   Take $\frac{d}{dx}$ as before: $-\frac{dy}{dx} * \sin(y) = 1 + \frac{dy}{dx}$. So
   $$\frac{dy}{dx}(-\sin(y) - 1) = 1, \qquad \text{and so } \boxed{\frac{dy}{dx} = \frac{1}{-\sin(y) - 1}}$$
2. Calculate $\frac{dx}{dy}$
   Now take $\frac{d}{dy}$: $-\sin(y) = \frac{dx}{dy} + 1$. So
   $$\boxed{\frac{dx}{dy} = -\sin(x) - 1}$$

Notice:

$$\frac{dy}{dx} = 1/\left(\frac{dx}{dy}\right)$$

This is true in general!

Using implicit differentiation for good:
Inverse functions.

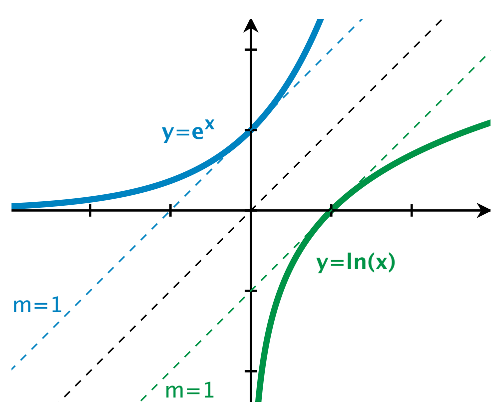
The Derivative of $y = \ln x$

Remember:

(1) $y = e^x$ has a slope through the point (0,1) of 1.

(2) The natural log is the *inverse* to e^x , so

$$y = \ln x \quad \Longrightarrow \quad e^y = x$$



The Derivative of $y = \ln x$

To find the derivative of $\ln(x)$, use implicit differentiation!
Rewrite

$$y = \ln x \quad \text{as} \quad e^y = x$$

Take a derivative of both sides of $e^y = x$ to get

$$\frac{dy}{dx} e^y = 1 \qquad \text{so} \qquad \frac{dy}{dx} = \frac{1}{e^y}$$

Problem: We asked "what is the derivative of $\ln(x)$?" and got back and answer with y in it!
Solution: Substitute back!

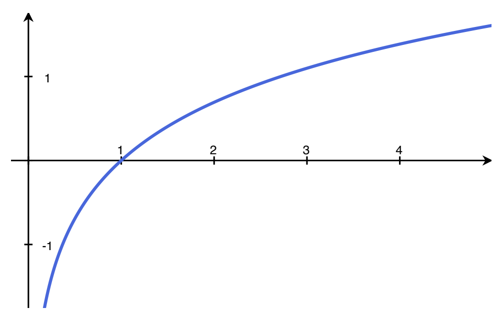
$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$$

$$\boxed{\frac{d}{dx}\ln(x) = \frac{1}{x}}$$

Does it make sense?

$$\boxed{\frac{d}{dx}\ln(x) = \frac{1}{x}}$$

$f(x) = \ln(x)$



$f(x) = \frac{1}{x}$

Examples

Calculate

1. $\frac{d}{dx} \ln x^2$

2. $\frac{d}{dx} \ln(\sin(x^2))$

3. $\frac{d}{dx} \log_3(x)$

   [hint: $\log_a x = \frac{\ln x}{\ln a}$]

Back to inverses

In the case where $y = \ln(x)$, we used the fact that $\ln(x) = f^{-1}(x)$, where $f(x) = e^x$, and got

$$\frac{d}{dx}\ln(x) = \frac{1}{e^{\ln(x)}}.$$

In general, calculating $\frac{d}{dx}f^{-1}(x)$:

(1) Rewrite $y = f^{-1}(x)$ as $f(y) = x$.

(2) Use implicit differentiation:

$$f'(y) * \frac{dy}{dx} = 1 \qquad \text{so} \qquad \boxed{\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}}.$$

Examples

Just to check, use the rule

$$\boxed{\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}}$$

to calculate

1. $\frac{d}{dx}\ln(x)$ (the inverse of e^x)
   In the notation above, $f^{-1}(x) = \ln(x)$ and $f(x) = e^x$.
   We'll also need $f'(x) = e^x$. So
   $$\boxed{\frac{d}{dx}\ln(x) = \frac{1}{e^{\ln(x)}}} \quad ☺$$
2. $\frac{d}{dx}\sqrt{x}$ (the inverse of x^2)
   In the notation above, $f^{-1}(x) = \sqrt{x}$ and $f(x) = x^2$.
   We'll also need $f'(x) = 2x$. So
   $$\boxed{\frac{d}{dx}\sqrt{x} = \frac{1}{2 * (\sqrt{x})}} \quad ☺$$

Inverse trig functions

Two notations:

$f(x)$	$f^{-1}(x)$
$\sin(x)$	$\sin^{-1}(x) = \arcsin(x)$
$\cos(x)$	$\cos^{-1}(x) = \arccos(x)$
$\tan(x)$	$\tan^{-1}(x) = \arctan(x)$
$\sec(x)$	$\sec^{-1}(x) = \text{arcsec}(x)$
$\csc(x)$	$\csc^{-1}(x) = \text{arccsc}(x)$
$\cot(x)$	$\cot^{-1}(x) = \text{arccot}(x)$

There are lots of points we know on these functions...

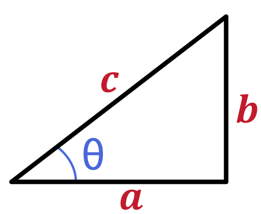
Examples:

1. Since $\sin(\pi/2) = 1$, we have $\arcsin(1) = \pi/2$
2. Since $\cos(\pi/2) = 0$, we have $\arccos(0) = \pi/2$

Etc...

In general:

arc___(-) takes in a ratio and spits out an angle:



$\cos(\theta) = a/c$ so $\arccos(a/c) = \theta$

$\sin(\theta) = b/c$ so $\arcsin(b/c) = \theta$

$\tan(\theta) = b/a$ so $\arctan(b/a) = \theta$

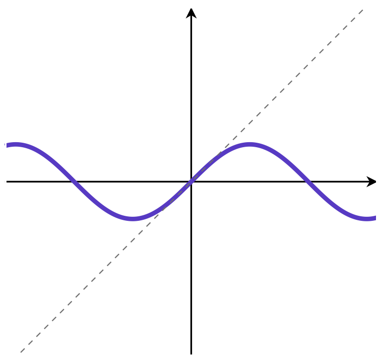
Domain problems:

$\sin(0) = 0, \quad \sin(\pi) = 0, \quad \sin(2\pi) = 0, \quad \sin(3\pi) = 0, \ldots$

So which is the right answer to $\arcsin(0)$, really?

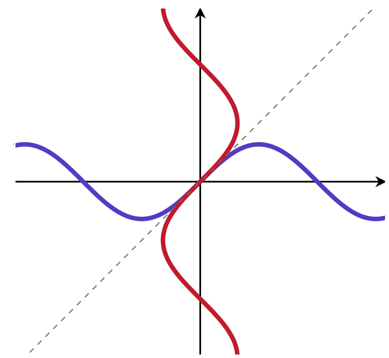
Domain/range

$y = \sin(x)$



Domain/range

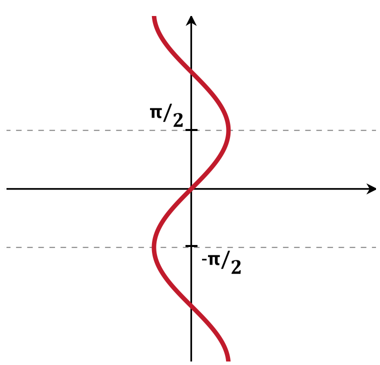
$y = \sin(x)$
$y = \arcsin(x)$



Domain: $-1 \leq x \leq 1$

Domain/range

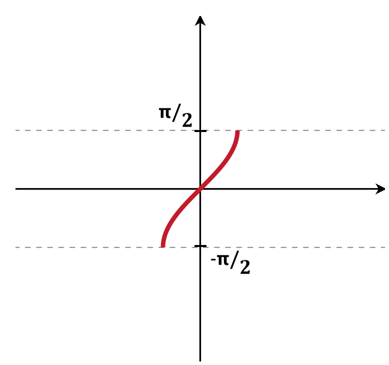
$y = \arcsin(x)$



Domain: $-1 \leq x \leq 1$

Domain/range

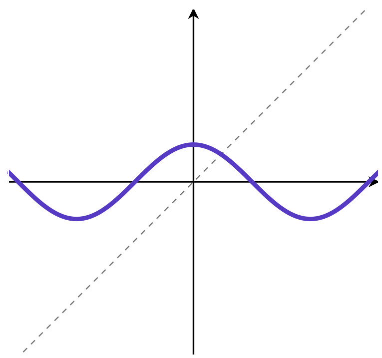
$y = \arcsin(x)$



Domain: $-1 \leq x \leq 1$ Range: $-\pi/2 \leq y \leq \pi/2$

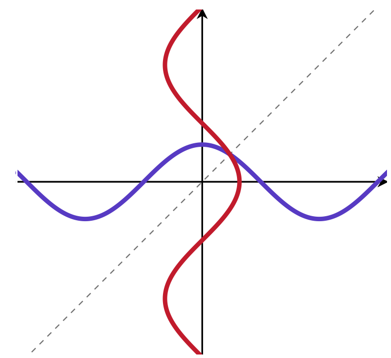
Domain/range

$y = \cos(x)$



Domain/range

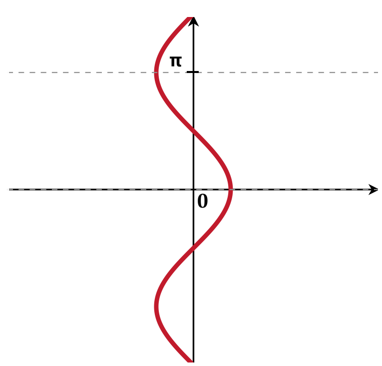
$y = \cos(x)$
$y = \arccos(x)$



Domain: $-1 \leq x \leq 1$

Domain/range

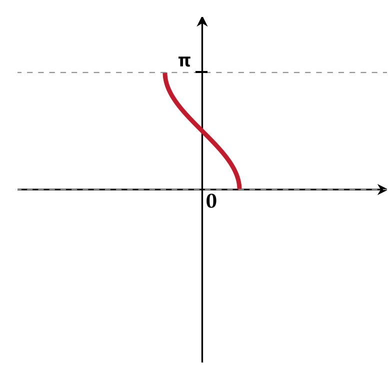
$y = \arccos(x)$



Domain: $-1 \leq x \leq 1$

Domain/range

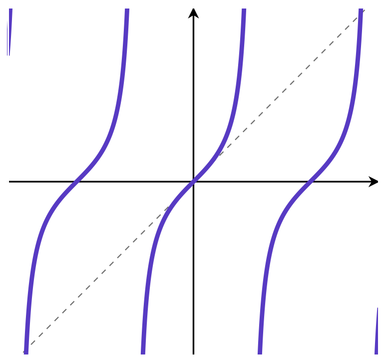
$y = \arccos(x)$



Domain: $-1 \leq x \leq 1$ Range: $0 \leq y \leq \pi$

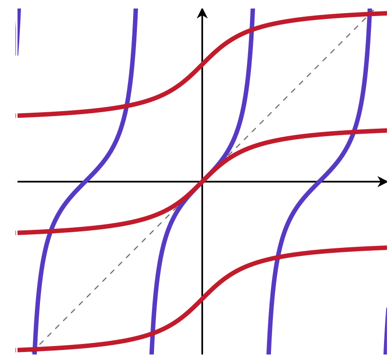
Domain/range

$y = \tan(x)$



Domain/range

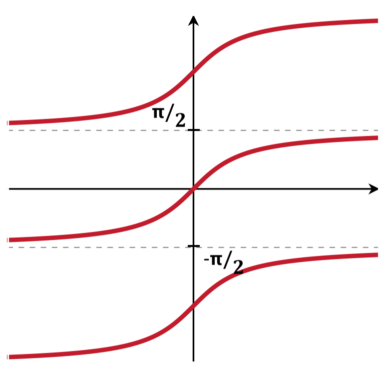
$y = \tan(x)$
$y = \arctan(x)$



Domain: $-\infty \leq x \leq \infty$

Domain/range

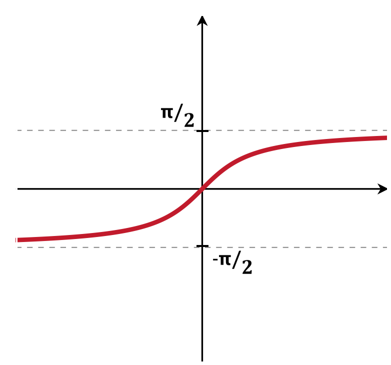
$y = \arctan(x)$



Domain: $-\infty \leq x \leq \infty$

Domain/range

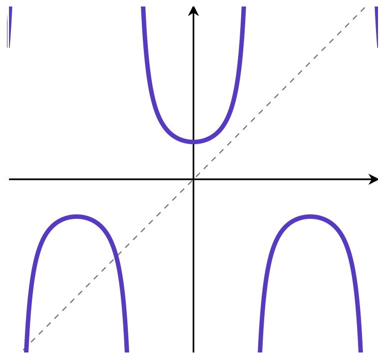
$y = \arctan(x)$



Domain: $-\infty \leq x \leq \infty$ Range: $-\pi/2 < y < \pi/2$

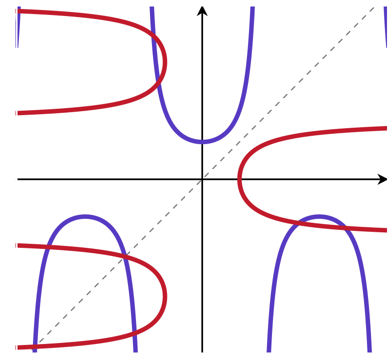
Domain/range

$y = \sec(x)$



Domain/range

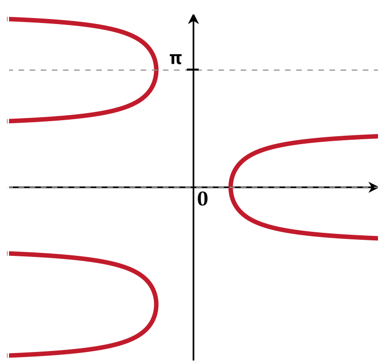
$y = \sec(x)$
$y = \text{arcsec}(x)$



Domain: $x \leq -1$ and $1 \leq x$

Domain/range

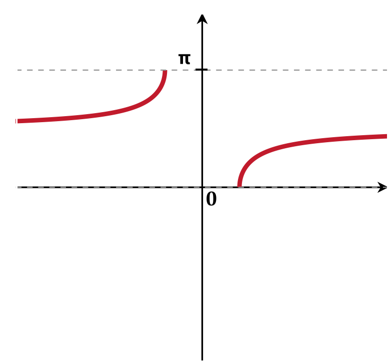
$y = \text{arcsec}(x)$



Domain: $x \leq -1$ and $1 \leq x$

Domain/range

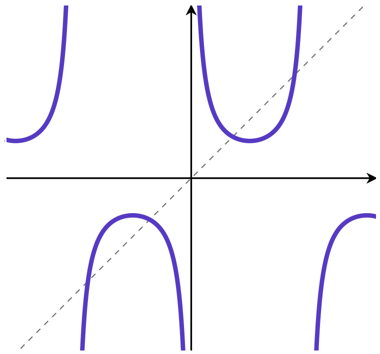
$y = \text{arcsec}(x)$



Domain: $x \leq -1$ and $1 \leq x$ Range: $0 \leq y \leq \pi$

Domain/range

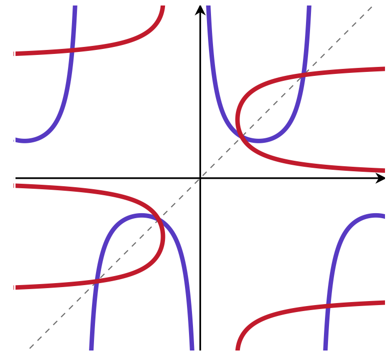
$y = \csc(x)$



Domain/range

$y = \csc(x)$

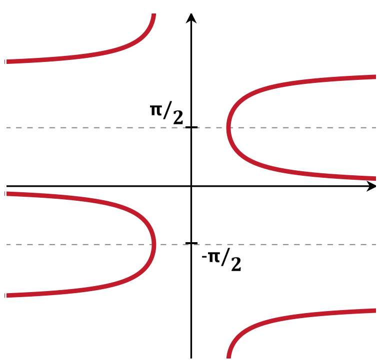
$y = \text{arccsc}(x)$



Domain: $x \leq -1$ and $1 \leq x$

Domain/range

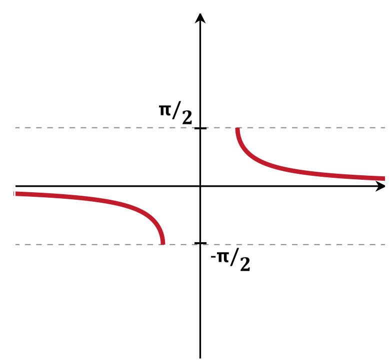
$y = \text{arccsc}(x)$



Domain: $x \leq -1$ and $1 \leq x$

Domain/range

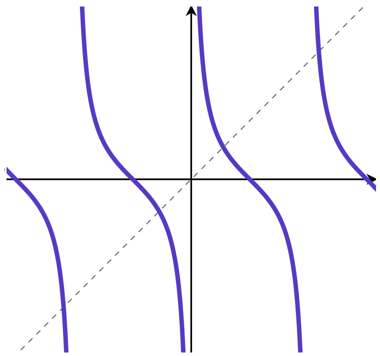
$y = \text{arccsc}(x)$



Domain: $x \leq -1$ and $1 \leq x$ Range: $-\pi/2 \leq y \leq \pi/2$

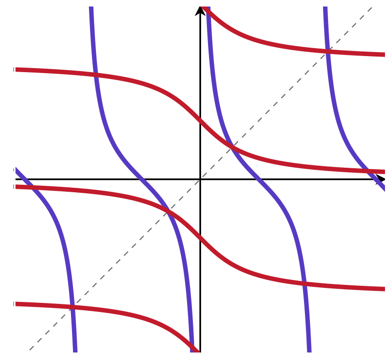
Domain/range

$y = \cot(x)$



Domain/range

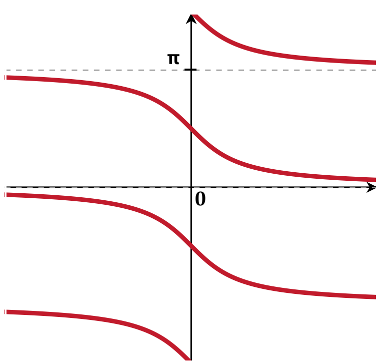
$y = \cot(x)$
$y = \text{arccot}(x)$



Domain: $-\infty \leq x \leq \infty$

Domain/range

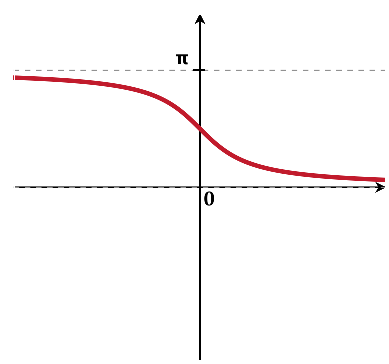
$y = \text{arccot}(x)$



Domain: $-\infty \leq x \leq \infty$

Domain/range

$y = \text{arccot}(x)$



Domain: $-\infty \leq x \leq \infty$ Range: $0 < y < \pi$

Back to Derivatives

Use implicit differentiation to calculate the derivatives of

1. $\arcsin(x)$
2. $\arctan(x)$

Use the rule

$$\boxed{\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}}$$

to check your answers, and then to calculate the derivatives of the other inverse trig functions:

1. $\frac{d}{dx} \arccos(x)$
2. $\frac{d}{dx} \text{arcsec}(x)$
3. $\frac{d}{dx} \text{arccsc}(x)$
4. $\frac{d}{dx} \text{arccot}(x)$

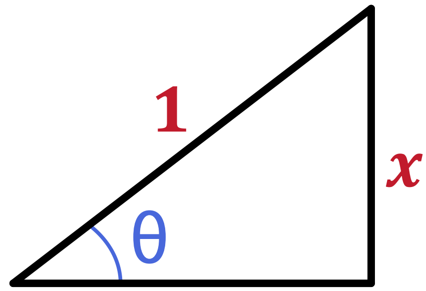
Recall:

$f(x)$	$f'(x)$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\csc(x)\cot(x)$
$\cot(x)$	$-\csc^2(x)$

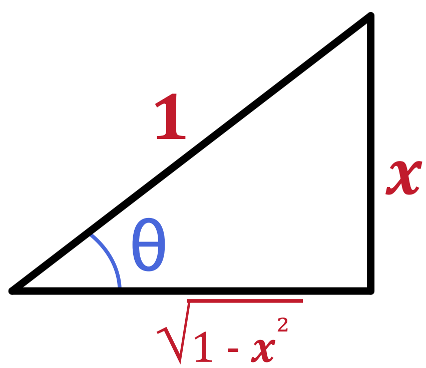
Simplifying $\cos(\arcsin(x))$

Call $\arcsin(x) = \theta$.

$\sin(\theta) = x$



Key: This is a simple triangle to write down whose angle θ has $\sin(\theta) = x$



So $\cos(\arcsin(x)) = \sqrt{1 - x^2}$

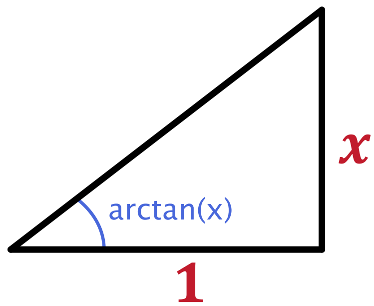
So $\dfrac{d}{dx} \arcsin(x) = \dfrac{1}{\cos(\arcsin(x))} = \dfrac{1}{\sqrt{1 - x^2}}$.

Calculating $\frac{d}{dx}\arctan(x)$.

We found that

$$\frac{d}{dx}\arctan(x) = \frac{1}{\sec^2(x)} = \left(\frac{1}{\sec(x)}\right)^2$$

Simplify this expression using



To simplify the rest, use the triangles