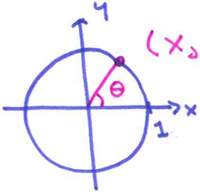
Derivatives of the Trig and Exponential Functions

Trig Identities:



$(x, y) = (\cos\theta, \sin\theta)$

Resulting identities:

$\cos(-\theta) = \cos\theta$

$\sin(-\theta) = -\sin\theta$

$\cos^2(\theta) + \sin^2\theta = 1$

Other useful identities:

$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$

The derivative of sine

$$\frac{d}{dx}\sin x =$$

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$$= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}$$

The derivative of sine

$$
\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)(\cos(h) - 1) + \cos(x)\sin(h)}{h}
\end{aligned}
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&= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h} \\
&= \sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}
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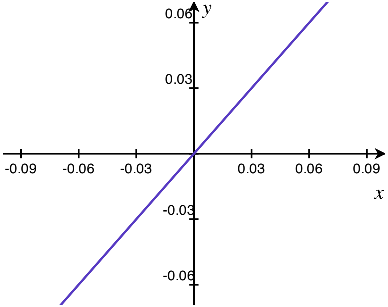
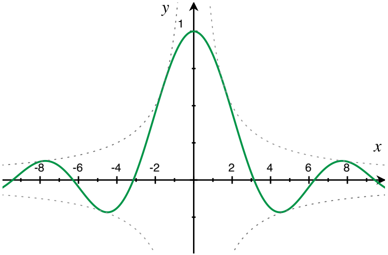
&= \sin(x)\lim_{h\to 0} \frac{\cos(h)-1}{h} + \cos(x)\lim_{h\to 0} \frac{\sin(h)}{h}
\end{aligned}$$

Recall: $\cos(0) = 1$ and $\sin(0) = 0$

Near $x = 0$, $\sin(x) \approx x$:

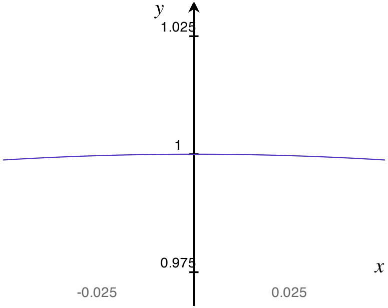


Graph of $\frac{\sin(x)}{x}$:

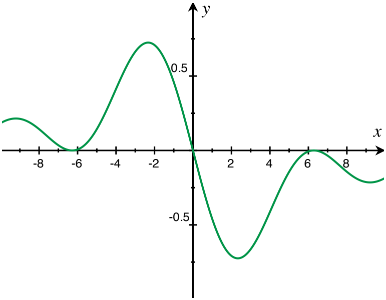


$$\boxed{\lim_{x \to 0} \frac{\sin(x)}{x} = 1}$$

Near $x = 0$, $\cos(x) \approx 1$:



Graph of $\frac{\cos(x)-1}{x}$:



$$\lim_{x\to 0} \frac{\cos(x) - 1}{x} = 0$$

The derivative of sine

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}\\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}\\
&= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h}\\
&= \sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}
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&= \sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h} \\
&= \sin(x)*0+\cos(x)*1
\end{aligned}
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&= \boxed{\cos(x)}
\end{aligned}
$$

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\begin{aligned}
\frac{d}{dx}\cos x &= \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x)(\cos(h) - 1) - \sin(x)\sin(h)}{h}
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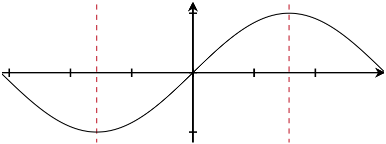
&= \lim_{h\to 0} \frac{\cos(x)(\cos(h) - 1) - \sin(x)\sin(h)}{h} \\
&= \cos(x) \lim_{h\to 0} \frac{\cos(h) - 1}{h} - \sin(x) \lim_{h\to 0} \frac{\sin(h)}{h} \\
&= \cos(x) * 0 - \sin(x) * 1 \\
&= \boxed{-\sin(x)}
\end{aligned}$$

Does it make sense?

$y = \sin(x)$:



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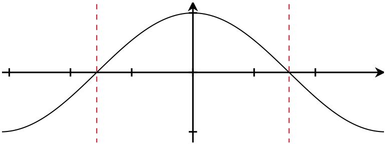
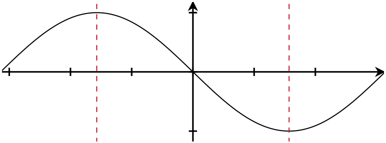


$y = -\sin(x)$:



Examples

Calculate...

1. $\frac{d}{dx} \sin(2x)$

2. $\frac{d}{dx} \cos(3x + \sqrt{x})$

3. $\frac{d}{dx} \sin(x)\cos(x)$

   Notice: $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$, and $\cos^2(x) - \sin^2(x) = \cos(2x)$.

   Does your answer still make sense from this perspective?

4. $\frac{d}{dx} \sin(\cos(x^2 + 2))$

On your own, fill in the rest of the trig functions:

1. $\frac{d}{dx}\tan(x)$

2. $\frac{d}{dx}\cot(x)$

3. $\frac{d}{dx}\sec(x)$

4. $\frac{d}{dx}\csc(x)$

On your own, fill in the rest of the trig functions:

1. $\frac{d}{dx}\tan(x) = \frac{d}{dx}\frac{\sin(x)}{\cos(x)}$

2. $\frac{d}{dx}\cot(x) = \frac{d}{dx}\frac{\cos(x)}{\sin(x)}$

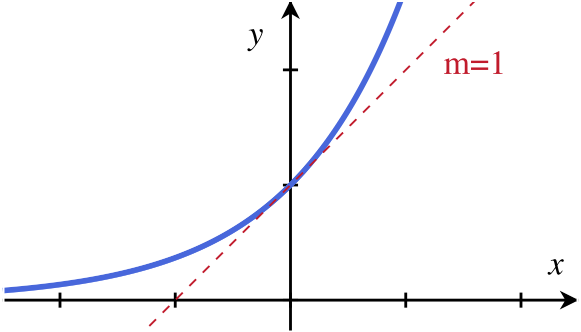
3. $\frac{d}{dx}\sec(x) = \frac{d}{dx}(\cos(x))^{-1}$

4. $\frac{d}{dx}\csc(x) = \frac{d}{dx}(\sin(x))^{-1}$

The Derivative of $y = e^x$

Recall!

e^x is the unique exponential function whose slope at $x = 0$ is 1:



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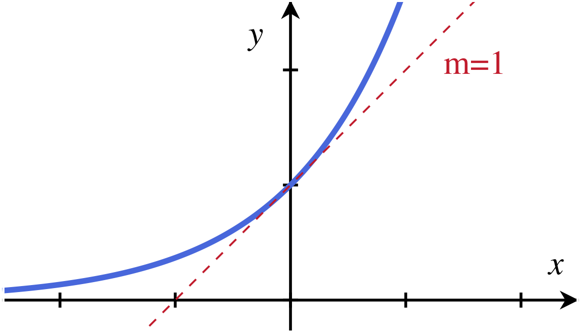

$$\lim_{h \to 0} \frac{e^{0+h} - e^0}{h} = \boxed{\lim_{h \to 0} \frac{e^h - 1}{h} = 1}$$

The Derivative of $y = e^x$...

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$$\begin{aligned} \frac{d}{dx} e^x &= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \\ &= \lim_{h \to 0} \frac{e^x \left(e^h - 1\right)}{h} \end{aligned}$$

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\end{aligned}$$

So

$$\boxed{\frac{d}{dx} e^x = e^x}$$

Examples

Calculate...

1. $\frac{d}{dx} e^{17x}$

2. $\frac{d}{dx} e^{x \ln(3)}$

3. $\frac{d}{dx} e^{\sin x}$

4. $\frac{d}{dx} e^{\sqrt{x^2+x}}$

Other bases

We know $\frac{d}{dx}x^a = ax^{a-1}$, but what is $\frac{d}{dx}a^x$?

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So

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln(a)} = \ln(a)e^{x\ln(a)}$$

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Example: $\frac{d}{dx}2^x = \ln(2) * 2^x$

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Sanity check: $\frac{d}{dx}e^x \stackrel{?}{=} \ln(e) * e^x$

Other bases

We know $\frac{d}{dx}x^a = ax^{a-1}$, but what is $\frac{d}{dx}a^x$?
Notice:

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So

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln(a)} = \ln(a)e^{x\ln(a)} = \ln(a) * a^x$$

Example: $\frac{d}{dx}2^x = \ln(2) * 2^x$

Sanity check: $\frac{d}{dx}e^x \stackrel{?}{=} \ln(e) * e^x = 1 * e^x$ ☺

Derivative so far

Definition: $f'(x) \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Combining functions:

$$\frac{d}{dx} c * f(x) = c * f'(x) \qquad \frac{d}{dx}\left(f(x) + g(x)\right) = f'(x) + g'(x)$$

$$\frac{d}{dx}\left(f(x) * g(x)\right) = f(x)g'(x) + g(x)f'(x) \qquad \frac{d}{dx}(f(g(x)) = f'(g(x)) * g'(x)$$

Basic functions:

$f(x)$	x^a	$\sin(x)$	$\cos(x)$	e^x
$f'(x)$	ax^{a-1}	$\cos(x)$	$-\sin(x)$	e^x

Other functions:

$f(x)$	$\tan(x)$	$\cot(x)$	$\sec(x)$	$\csc(x)$	a^x
$f'(x)$	$\sec^2(x)$	$-\csc^2(x)$	$\sec(x)\tan(x)$	$-\csc(x)\cot(x)$	$\ln(a)a^x$

Example

Compute the derivatives of

1. $e^{17x^2+\sqrt{x}}$
2. $\tan\left(e^{17x^2+\sqrt{x}}\right)$
3. $\csc(x) * \left[3^x + \tan^3\left(e^{17x^2+\sqrt{x}}\right)\right]^4$