Derivatives

Definition

The *derivative* of a function f is a new function defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

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Q. How is this the same or different from what we were doing yesterday with tangent lines?

A. Yesterday, we were calculating derivatives at individual points, and getting *numbers* for answers. Today, we'll calculate the *derivative function*, and get out answers with variables in them (do all the points at once).

Example: let $f(x) = x^2$

Derivatives at a point: If I first ask "what is $f'(2)$?", I could calculate

$$f'(2) = \lim_{h \to 0} \frac{(2+h)^2 - 2^2}{h} = \lim_{h \to 0} \frac{4h + h^2}{h} = \lim_{h \to 0} 4 + h = \boxed{4}.$$

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Today's goal: Write down a *function* $f'(x)$ which has all the derivatives-at-a-point collected together.

If a is a number, (like 2 or 3) then

$$f'(a) = \underbrace{\lim_{h\to 0}}_{\substack{\text{gets rid} \\ \text{of the } h\text{'s}}} \underbrace{\frac{f(a+h)-f(a)}{h}}_{\substack{\text{a function of} \\ a\text{'s and } h\text{'s}}}$$

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Starting simple

Suppose we consider the piecewise linear function

$$f(x) = \begin{cases} x & x \leq 1 \\ 1 & 1 < x < 3 \\ -x + 4 & 3 \leq x \end{cases}$$

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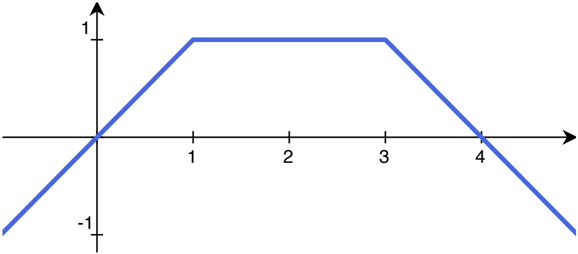

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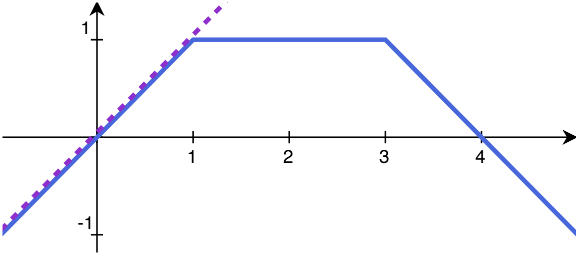

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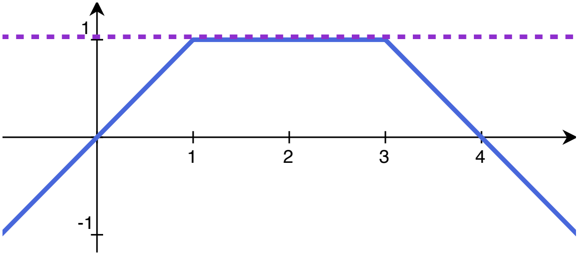

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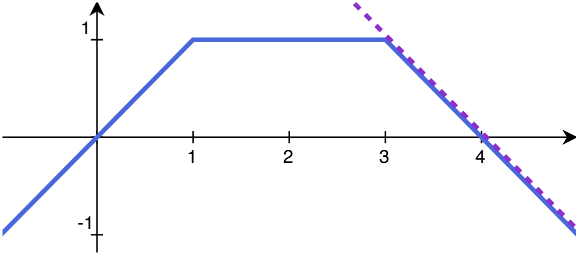

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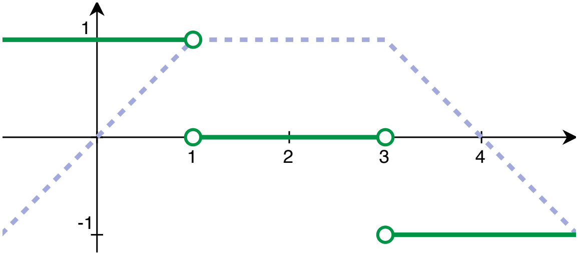


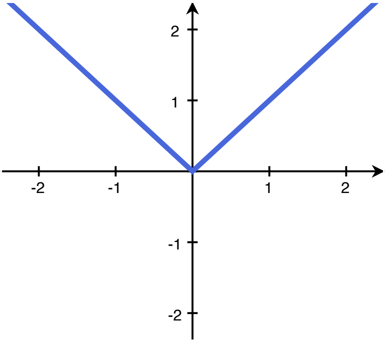
The derivative is:

$$f(x) = \begin{cases} 1 & x < 1 \\ 0 & 1 < x < 3 \\ -1 & 3 < x \end{cases}$$

Another example

What is the derivative of $f(x) = |x|$?

Write down the piecewise function and sketch it on the graph.



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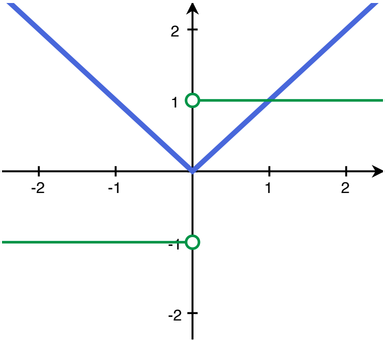
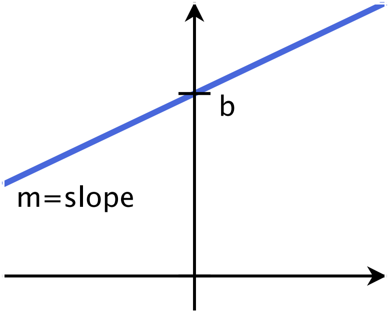


$$f'(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$

Lines

In general, if m and b are constants, and

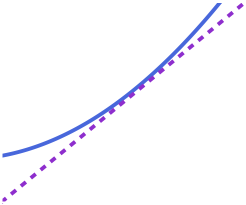
$$f(x) = mx + b$$



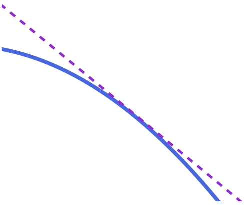
$$f'(x) = m$$

the slope of the tangent line = slope of the line

Rough shape of the derivative



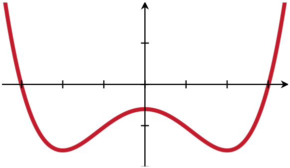
if $f(x)$ is increasing
$f'(x)$ is positive!

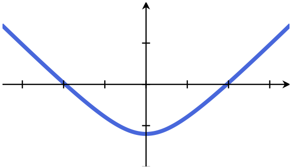


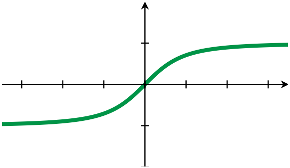
if $f(x)$ is decreasing
$f'(x)$ is negative!

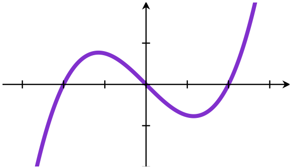
Match em up!

Here are graphs of two functions and their derivatives. Which are which?









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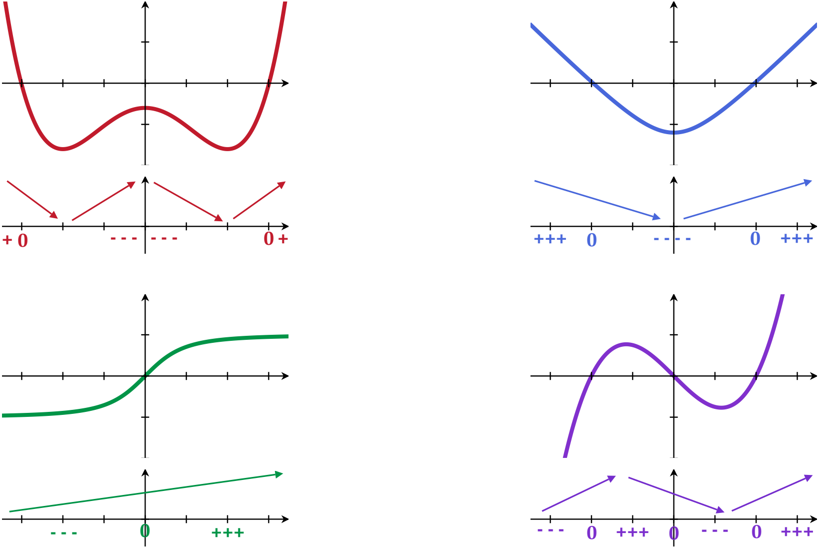

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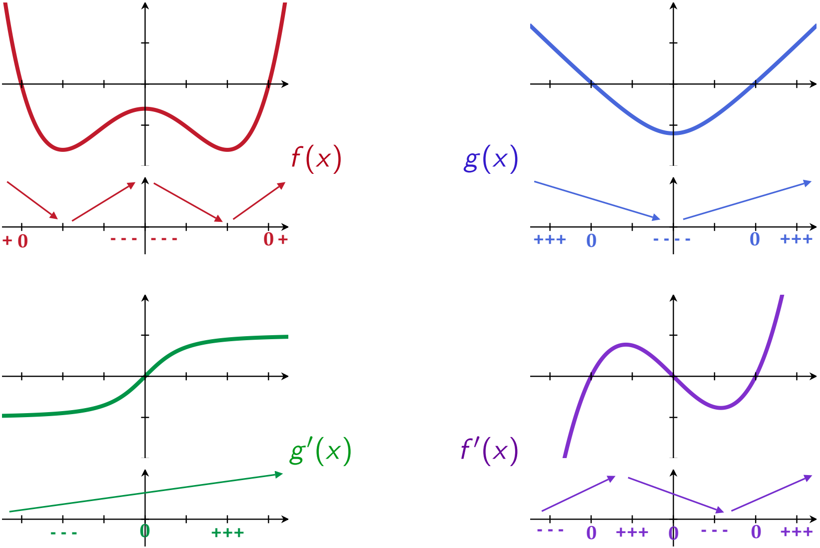

A little more notation

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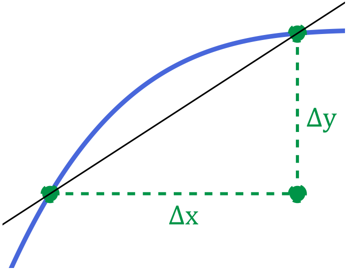

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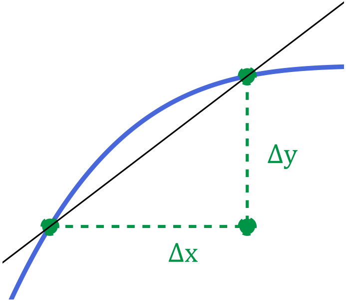

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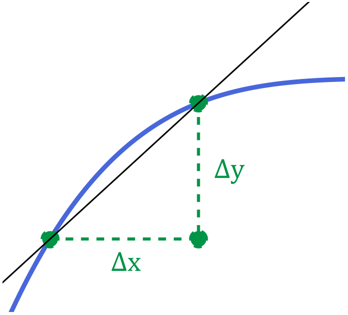

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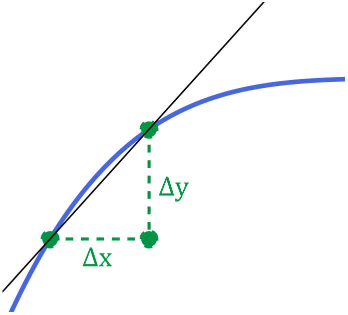

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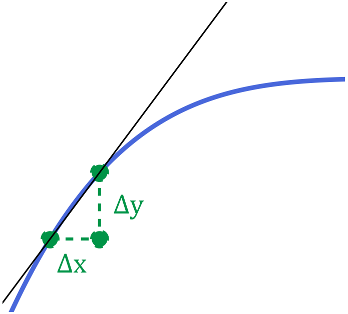

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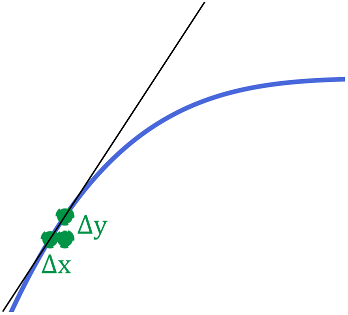

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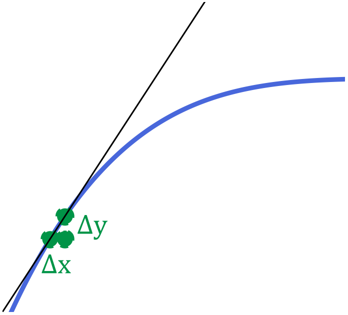

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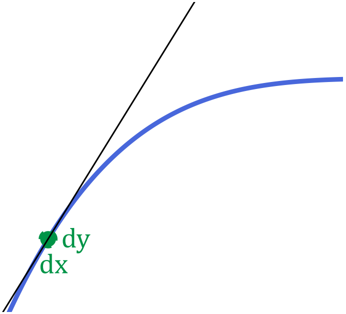

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$$\Delta x \rightsquigarrow dx \qquad \Delta y \rightsquigarrow dy$$

$$\frac{\Delta y}{\Delta x} \rightsquigarrow \frac{dy}{dx}$$

"infinitesimals"

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Leibniz notation

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Example: We can write the derivative of x^2 as $\frac{d}{dx} x^2$

and the derivative of x^2 at $x = 5$ as $\left.\frac{d}{dx} x^2\right|_{x=5}$

Go to work: building our first derivative rule.

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By taking limits, fill in the rest of the table:

$f(x)$	1	x	x^2	x^3	$\frac{1}{x}$	$\frac{1}{x^2}$	$\sqrt{x}$	$\sqrt[3]{x}$
$f'(x)$			$2x$					

Hints: For $\frac{1}{x^2}$, find a common denominator, and then expand.

For $\sqrt[3]{x}$, try multiplying and dividing by $(\sqrt[3]{x+h})^2+(\sqrt[3]{x+h})(\sqrt[3]{x})+(\sqrt[3]{x})^2$.

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$f'(x)$	0	1	$2x$	$3x^2$	$-\frac{1}{x^2}$	$-\frac{2}{x^3}$	$\frac{1}{2\sqrt{x}}$	$\frac{1}{3(\sqrt[3]{x})^2}$

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$f(x)$	$f'(x)$
$x^0 = 1$	0
$x^1 = x$	1
x^2	$2x$
x^3	$3x^2$
$\frac{1}{x} = x^{-1}$	$-x^{-2}$
$\frac{1}{x^2} = x^{-2}$	$-2x^{-3}$
$\sqrt{x} = x^{1/2}$	$(1/2)x^{-1/2}$
$\sqrt[3]{x} = x^{1/3}$	$(1/3)x^{-2/3}$

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Power rule:

$$\frac{d}{dx}x^a = ax^{a-1}$$

Use the power rule to take consecutive derivatives of $x^{5/2}$:

$$x^{5/2} \xrightarrow{\frac{d}{dx}} \boxed{\quad\quad}$$

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Use the power rule to take consecutive derivatives of $x^{5/2}$:

$$x^{5/2} \xrightarrow{\frac{d}{dx}} \boxed{\tfrac{5}{2}x^{3/2}}$$

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$$\xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2} * \frac{3}{2}x^{1/2}}$$

$$\xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2} * \frac{3}{2} * \frac{1}{2}x^{-1/2}}$$

$$\xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2} * \frac{3}{2} * \frac{1}{2} * \left(-\frac{1}{2}\right) x^{-3/2}}$$

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$$x^{5/2} \xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2}x^{3/2}}$$ *1st derivative*

$$\xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2} * \frac{3}{2}x^{1/2}}$$ *2nd derivative*

$$\xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2} * \frac{3}{2} * \frac{1}{2}x^{-1/2}}$$ *3rd derivative*

$$\xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2} * \frac{3}{2} * \frac{1}{2} * \left(-\frac{1}{2}\right) x^{-3/2}}$$

4th derivative

Use the power rule to take consecutive derivatives of $x^{5/2}$:

$$x^{5/2} \xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2}x^{3/2}} \quad \textit{1}^{st} \textit{ derivative} = f'(x) = \frac{d}{dx}x^2$$

$$\xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2} * \frac{3}{2}x^{1/2}} \quad 2^{nd} \textit{ derivative} = f''(x) = \frac{d^2}{dx^2}x^2$$

$$\xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2} * \frac{3}{2} * \frac{1}{2}x^{-1/2}} \quad 3^{rd} \textit{ derivative} = f^{(3)}(x) = \frac{d^3}{dx^3}x^2$$

$$\xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2} * \frac{3}{2} * \frac{1}{2} * \left(-\frac{1}{2}\right) x^{-3/2}}$$

$$4^{th} \textit{ derivative} = f^{(4)}(x) = \frac{d^4}{dx^4}x^2$$

Use the power rule to take consecutive derivatives of $x^{5/2}$:

$$x^{5/2} \xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2}x^{3/2}} \quad 1^{st} \text{ derivative} = f'(x) = \frac{d}{dx}x^2$$

$$\xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2} * \frac{3}{2}x^{1/2}} \quad 2^{nd} \text{ derivative} = f''(x) = \frac{d^2}{dx^2}x^2$$

$$\xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2} * \frac{3}{2} * \frac{1}{2}x^{-1/2}} \quad 3^{rd} \text{ derivative} = f^{(3)}(x) = \frac{d^3}{dx^3}x^2$$

$$\xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2} * \frac{3}{2} * \frac{1}{2} * \left(-\frac{1}{2}\right) x^{-3/2}}$$

$$4^{th} \text{ derivative} = f^{(4)}(x) = \frac{d^4}{dx^4}x^2$$

Definition: The n^{th} derivative of $f(x)$ is

$$\underbrace{\frac{d}{dx}\frac{d}{dx}\cdots\frac{d}{dx}}_{n} f(x) = \frac{d^n}{dx^n}f(x) = f^{(n)}(x).$$